

Jordan Block Based Polynomial Collocation Technique for the Numerical Solution of Two Dimensional PVIDEs

Ojo O. Aduroja¹, Samuel Adamu^{*2} & Albert Ayuba Shalangwa³

¹Department of Mathematics, University of Ilesa, Ilesa, Osun State, Nigeria

²Department of Mathematics, Nigerian Army University Biu, Borno State, Nigeria

³Department of Mathematical Sciences, Gombe State University, Gombe State, Nigeria

*Corresponding Author's Email: malgwisa@gmail.com (Adamu, S.)

Received on: January, 2026

Revised and Accepted on: February, 2026

Published on: March, 2026

ABSTRACT

The paper present a polynomial method based on the Jordan block operational matrix for the solution of 2-dimensional partial Volterra Integro Differential Equations (VIDEs). The technique transforms the partial VIDEs to integral equation, this is then approximated through polynomial collocation at standard points, leading to a system of algebraic equations. It establishes the uniqueness of the solution by applying Banach fixed-point theory. To demonstrate the accuracy and reliability of the method, linear and nonlinear mathematical problems established in literature with known exact solutions are solved with the aid of a written MATLAB code. The results obtained prove that the technique is capable of reproducing the exact solutions with zero absolute error when the exact solution lies within the chosen polynomial space. The computational results further indicate rapid convergence and low computational cost, highlighting the spectral-type accuracy of the method. Hence, the Jordan block polynomial collocation technique provides a simple, accurate, and computationally efficient numerical technique for solving 2-dimensional partial VIDEs.

Keywords: Collocation technique, Polynomial collocation, Jordan block, Banach fixed point theorem

1.0 INTRODUCTION

Functional equations, such as Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs), integral and integro differential equations, are typically produced by mathematical models of real-world issues (Noori & Taghizadeh, 2020; Adamu *et al.*, 2023). In 1908, Vito Volterra investigated the effects of heredity while analyzing a population growth model. The study

produced a model in which integral and differential operators coexisted in the model equation (Osilagun *et al.*, 2023; Shalangwa *et al.*, 2025), which is termed Integro-Differential Equations (IDEs).

This study consider Partial VIDEs in 2D-spatial domains with one dimension of time in the form

$$\frac{\partial u(x,t)}{\partial t} = g(x,t) + \int_0^x \int_0^t k(t,s,x,y)u(s,y)dyds, \quad x \in [0,1], t \in [0,1] \quad (1)$$

and the initial condition

$$u(x,0) = u_0(x) \quad (2)$$

where $k(t,s,x,y)$ is the kernel of the integral equation and $g(x,t)$ are given functions on $[0,1]^2$. $u(x,t)$ is the unknown function which is a function of two variables x and t where x is the space variable and t is the time variable. An essential model for simulating the impacts of the system's "memory" is provided by parabolic partial VIDEs. Equation (1) is a partial differential equation that results in parabolic partial VIDEs by combining the integral term with the unknown kernel function and partial differentiation (Zhao & Zhao, 2016; Adamu *et al.*, 2025).

Many branches of science, engineering, and the social sciences naturally use partial integro-differential equations (PIDE) (Burcu & Mehmet, 2017; Adamu *et al.*, 2025). PVIDEs of the form (1) originate from different

physical and biological models as well as mathematical modeling of the spatiotemporal development of an epidemic (Pachpatte, 2011; Shalangwa, Odekunle & Ade, 2025). Heat conduction in materials, compression of poro viscoelastic media, nuclear reactor dynamics (Zhao & Zhao, 2016; Burcu & Mehmet, 2017; Adesanya *et al.*, 2023), thermo-elastic contact, control theory, fluid dynamics, and viscoelastic mechanics (Aziz & Khan, 2017), blow-up problems, and reaction diffusion problems (Zhao & Zhao, 2016).

Otaide and Oluwayemi (2024) employed fourth-kind Chebyshev polynomials combined with a variational collocation algorithm to solve Volterra integro-differential equations, while Eashel *et al.* (2025) applied multi-step collocation techniques to obtain numerical solutions of VIDEs with non-vanishing delay. In a related spectral framework, Tedjani *et al.* (2025) utilized shifted Jacobi polynomials as basis functions for a spectral collocation method to solve high-order one- and two-dimensional VIDEs with variable coefficients.

Meanwhile, Mahdy et al. (2023) adopted the separation of variables technique to reduce two-dimensional mixed-dimensional difference integro-differential equations (2D-MDeIDEs) to one-dimensional Fredholm difference integro-differential equations, and subsequently applied the Bernoulli polynomial method to derive a system of linear algebraic equations for solving the resulting (1+1)-dimensional problems under mixed conditions.

ODEs and PDEs can be approximated using collocation methods (Adamu *et al.*, 2023). The principle of collocation method is to approximate the exact solutions $u(x,t)$ by determining an appropriate linear combination of a group of linearly independent basis functions $\{\varphi_1 \ \varphi_2 \ \varphi_3 \ \dots \ \varphi_k\}$ and

$\{\gamma_1 \ \gamma_2 \ \gamma_3 \ \dots \ \gamma_m\}$

$$u(x,t) \approx \sum_{i=0}^n \sum_{j=0}^m C_{ij} \varphi_i(x) \gamma_j(t) \quad (3)$$

such that $u(x,t)$ satisfies the given PDE at a number of appropriately located points (called collocation points), and at the same time, satisfies the initial or boundary conditions. The number of collocation points and the boundary points where the initial or boundary condition is satisfied determine the dimensions k and m of the polynomial space (Chen, 2017). Numerous numerical techniques, including collocation, Taylor-series expansion, quadratures, Chebyshev polynomial, sine-cosine wavelets, and Haar functions is used for solving the 1-dimensional partial VIDEs.

However, only a small amount of work has been done on partial VIDEs in 2-dimensional cases. Such methods include the Laplace transform (Zhao & Zhao, 2016), Euler Backward finite difference method (Adamu *et al.*, 2025), variation iterative method (Hussain, Fadhel, Rusli & Yahya, 2020), and finite difference method (Sameeh & Elsaid, 2016). Therefore, in this study, we will use the polynomial collocation technique to solve 2-dimensional partial VIDEs. Methods for solving two-Dimensional integro differential equations are profered by Hussain *et al.*, (2020), Khajehnasiri (2015) and Adamu *et al.*, (2025) but still the accuracy of the the solution need to be improved.

By using the polynomial collocation method to solve partial Volterra integro differential equations, this study add a new dimension to the use of collocation method. Because of their rapid rate of convergence, high accuracy, ease of computer programming, and suitability for complicated problems, polynomial collocation methods outperform the conventional finite-difference approach in numerical differentiations.

2.0 PRELIMINARIES

2.1 Convergence of the Collocation Approximation

Let $u(x,t)$ denote the exact solution of the two-dimensional partial VIDE on $\Omega = [a,b] \times [0,T]$, and let $u_N(x,t)$ be the polynomial collocation approximation of degree N constructed on Ω . Assume that $u, u_N \in C(\Omega)$. Define the error function by

$$e_N(x,t) = u_N(x,t) - u(x,t), (x,t) \in \Omega.$$

Suppose the Volterra kernel $K(x,t,\zeta,\tau)$ satisfies a Lipschitz condition with respect to its functional argument, that is,

$$|K(x,t,\zeta,\tau,u_1) - K(x,t,\zeta,\tau,u_2)| \leq L|u_1 - u_2|$$

for some constant $L > 0$. Then the sequence of collocation solutions $\{u_N\}_{\{N \geq 0\}}$ converges uniformly to the exact solution u on Ω .

2.2 Banach Fixed Point Framework for PVIDEs

Let $(X, \|\cdot\|_\infty)$ be the Banach space

$$X = C(\Omega),$$

$$\|v\|_\infty = \max_{(x,t) \in \Omega} |v(x,t)|.$$

Define the Volterra integral operator $F: X \rightarrow X$ associated with the PVIDE by

$$(Fu)(x,t) = g(x,t) + \int_0^t \int_a^x K(x,t,\xi,\tau) u(\xi,\tau) d\xi d\tau$$

If F is a contraction on X , then it admits a unique fixed point $u^* \in X$, which coincides with the exact solution of the PVIDE.

2.3 Uniqueness and Convergence of the Iterative Collocation Scheme

Assume that the operator $F: X \rightarrow X$ is a q -contraction with $0 < q < 1$. Then:

(i) The operator equation $u = Fu$ has a unique solution $u^* \in X$.

(ii) For any initial approximation $u_0 \in X$, the sequence generated by

$$u_{n+1} = Fu_n, \quad n = 0, 1, 2, \dots$$

converges uniformly to u^* on Ω .

(iii) The convergence rate satisfies the error estimate

$$\|u_n - u^*\|_\infty \leq \frac{q^n}{1-q} \|u_1 - u_0\|_\infty, \quad n \geq 1.$$

3.0 MATERIAL AND METHOD

3.1 Method of solution

Let the solution of equation (1) and equation (2) be approximated by polynomial approximate solution defined by equation (3), then

$$u(x, t) = \mathbf{X}(x)\mathbf{A}\mathbf{T}(t) \quad (4)$$

where

$$\mathbf{X}(x) = [1 \quad x \quad x^2 \quad \cdots \quad x^N], \quad \mathbf{T}(t) = [1 \quad t \quad t^2 \quad \cdots \quad t^M]^T,$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}^T$$

Substituting equation (4) into equation (1) result to

$$\mathbf{X}(x)\mathbf{A}\mathbf{T}(t)\mathbf{D} = g(x, t) + \int_0^x \int_0^t k(t, s, x, y)F(\mathbf{X}(s)\mathbf{A}\mathbf{T}(y)) dy ds \quad (5)$$

where

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

writing $g(x, t)$ in polynomial expansion

$$g(x, t) = \varphi(x)\mathbf{C}\varphi(t) \quad (6)$$

where $\mathbf{C} = \frac{1}{n!m!} \sum_{n=0}^N \sum_{m=0}^M \frac{\partial^{m+n} g(x, t)}{\partial x^m \partial t^n} \Big|_{x=x_0, t=t_0}$

Substituting equation (6) into equation (5)

$$\mathbf{X}(x)\mathbf{A}\mathbf{T}(t)\mathbf{D} = \varphi(x)\mathbf{C}\varphi(t) + \int_0^x \int_0^t k(t, s, x, y)F(\mathbf{X}(s)\mathbf{A}\mathbf{T}(y)) dy ds \quad (7)$$

Collocating equation (7) at x_i and t_j gives

$$\mathbf{X}(x_i)\mathbf{A}\mathbf{T}(t_j)\mathbf{D} - \varphi(x_i)\mathbf{C}\varphi(t_j) - \int_0^{x_i} \int_0^{t_j} k(t_j, s, x_i, y)F(\mathbf{X}(s)\mathbf{A}\mathbf{T}(y)) dy ds = 0 \quad (8)$$

Equation (8) gives $(N+1) \times (M+1)$ system of nonlinear equations which is then solved using Newton Raphson's method.

3.2 Analysis of the method

Definition (Function Space)

Let

$$X = C([0, 1] \times [0, 1])$$

Denote the space of all continuous functions on $[0, 1] \times [0, 1]$, equipped with the supremum norm

$$\|u\| = \sup_{(x,t) \in [0,1]^2} |u(x,t)|.$$

Then $(X, \|\cdot\|)$ is a Banach space.

Assumptions

We assume the following

(A1) $g(x, t)$ is continuous on $[0, 1] \times [0, 1]$;

(A2) $k(t, s, x, y)$ is continuous and bounded i.e

$$|k(t, s, x, y)| \leq K;$$

(A3) $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfies a Lipschitz condition

$$|F(u) - F(v)| \leq L|u - v|, \quad \forall u, v \in \mathbb{R}.$$

3.2.1 Equivalent Integral Form

Integrating equation (1) with respect to t from 0 to t and using the initial condition $u(x, 0) = u_0(x)$ yields

$$u(x, t) = u_0(x) + \int_0^t g(x, \tau) d\tau + \int_0^t \int_0^x \int_0^\tau k(\tau, s, x, y) F(u(s, y)) dy ds d\tau. \quad (9)$$

Definition

Define an operator $T: X \rightarrow X$ by

$$(Tu)(x, t) = u_0(x) + \int_0^t g(x, \tau) d\tau + \int_0^t \int_0^x \int_0^\tau k(\tau, s, x, y) F(u(s, y)) dy ds d\tau \quad (10)$$

Solving (1) – (2) is equivalent to finding a fixed point of T .

3.2.2 Existence of solution

Lemma 1 (Continuity of the operator)

The T is continuous on X .

Proof

Let $\{u_n\} \subset X$ with $u_n \rightarrow u$ uniformly.

By continuity of F , $F(u_n) \rightarrow F(u)$ uniformly.

Since g and k are continuous and bounded, the dominated convergence theorem ensures $Tu_n \rightarrow Tu$ uniformly.

Hence, T is continuous.

Lemma 2 (Compactness of the operator)

The operator T is compact on X .

Proof

The Volterra-type integral operator maps bounded sets in X into bounded and equi-continuous families. By the Arzila-Asoli theorem, such sets are relatively compact in X .

Thus, T is compact.

Theorem 1 (Existence of solution)

Under the assumption (A1) – (A3), the PVIDE (1) – (2) admits at least one continuous solution in X .

Proof

From Lemma 1 and Lemma 2, the operator T is continuous and compact. Moreover, T maps a closed, bounded, convex subset of X into itself. Hence, by Schauder's fixed point theorem, T has at least one fixed point $u \in X$. This fixed point is a solution of the PVIDE.

3.2.3 Uniqueness of Solution

Theorem 2 (Uniqueness)

If

$$\frac{KL}{2} < 1, \quad (11)$$

then the solution of the PVIDE (1) – (2) is unique.

Proof

Let $u, v \in X$ be two fixed points of T . Then

$$\begin{aligned} |Tu(x, t) - Tv(x, t)| &\leq \int_0^t \int_0^x \int_0^\tau |k(\tau, s, x, y)| |F(u) - F(v)| dy ds d\tau \\ &\leq KL \int_0^t \int_0^x \int_0^\tau |u - v| dy ds d\tau \end{aligned}$$

Evaluating the integral

$$\int_0^t \int_0^x \int_0^\tau dy ds d\tau = \frac{xt^2}{2} \leq \frac{1}{2}.$$

Hence

$$\|Tu - Tv\| \leq \frac{KL}{2} \|u - v\|$$

If condition (11) holds, T is a contraction. By the Banach fixed point theorem, the fixed point is unique.

4.0 RESULTS AND DISCUSSION

4.1 Numerical Experiments

In this section, numerical experiments are carried out to demonstrate the new concept's efficiency and accuracy. Let $u_N(t)$ and $u(t)$ be the approximate and numerical solution respectively, then $Error = |u_N(t) - u(t)|$ is the absolute error.

All computations in this section are done with the aid of a written MATLAB program. For all the examples, initial condition problems were considered.

Example (Hussain et al., 2020): Consider the linear partial integro differential two dimensional equation

$$\frac{\partial u(x, t)}{\partial t} = x + \frac{(t^2 x - tx^2)(tx + 3)}{6} + \int_0^x \int_0^t (s - y)u(s, y) dy ds, \quad (x, t) \in [0, 1]^2$$

with initial condition

$$u(x, 0) = 1, 0 \leq x \leq 1$$

and exact solution

$$u(x, t) = 1 + xt.$$

Solution Table 1 presents the numerical results for the approximate solution, exact solution and error for different values of x and t between 0 and 1. Figure 1, Figure 2 and Figure 3 give the comparative analysis of the results through visualization.

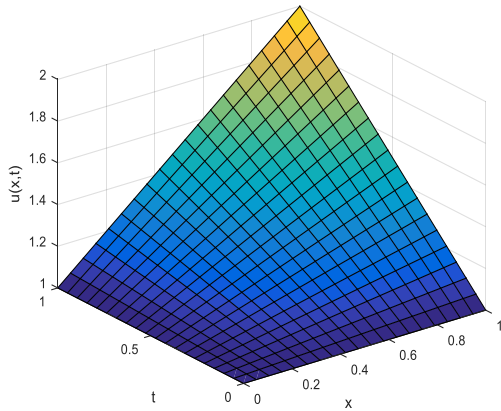


Figure 1: Numerical solution for example 1

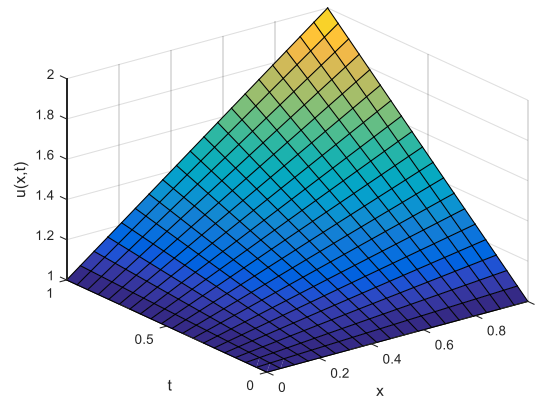


Figure 2: Exact solution for example 1

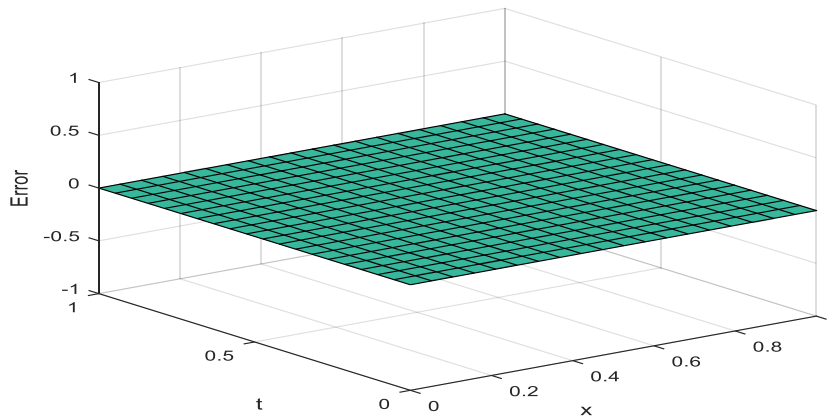


Figure 3: Absolute error for example 1

Table 1: Numerical values at selected points

x	t	Approximate solution	Hussain <i>et al.</i> , (202) Exact solution	Error	
0.00	0.00	1.000000e+00	1.000000e+00	1.000000e+00	0
0.00	0.25	1.000000e+00	1.000000e+00	1.000000e+00	0
0.00	0.50	1.000000e+00	1.000000e+00	1.000000e+00	0
0.00	0.75	1.000000e+00	1.000000e+00	1.000000e+00	0
0.00	1.00	1.000000e+00	1.000000e+00	1.000000e+00	0
0.25	0.00	1.000000e+00	1.000000e+00	1.000000e+00	0
0.25	0.25	1.062500e+00	1.062500e+00	1.062500e+00	0
0.25	0.50	1.125000e+00	1.125000e+00	1.125000e+00	0
0.25	0.75	1.187500e+00	1.187500e+00	1.187500e+00	0
0.25	1.00	1.250000e+00	1.250000e+00	1.250000e+00	0
0.50	0.00	1.000000e+00	1.000000e+00	1.000000e+00	0
0.50	0.25	1.125000e+00	1.125000e+00	1.125000e+00	0
0.50	0.50	1.250000e+00	1.250000e+00	1.250000e+00	0
0.50	0.75	1.375000e+00	1.375000e+00	1.375000e+00	0
0.50	1.00	1.500000e+00	1.500000e+00	1.500000e+00	0
0.75	0.00	1.000000e+00	1.000000e+00	1.000000e+00	0
0.75	0.25	1.187500e+00	1.187500e+00	1.187500e+00	0
0.75	0.50	1.375000e+00	1.375000e+00	1.375000e+00	0

0.75	0.75	1.562500e+00	1.562500e+00	1.562500e+00	0
0.75	1.00	1.750000e+00	1.750000e+00	1.750000e+00	0
1.00	0.00	1.000000e+00	1.000000e+00	1.000000e+00	0
1.00	0.25	1.250000e+00	1.250000e+00	1.250000e+00	0
1.00	0.50	1.500000e+00	1.500000e+00	1.500000e+00	0
1.00	0.75	1.750000e+00	1.750000e+00	1.750000e+00	0
1.00	1.00	2.000000e+00	2.000000e+00	2.000000e+00	0

Example (Hussain et al., 2020): Consider the nonlinear partial two dimensional integro differential equation

$$\frac{\partial u(x,t)}{\partial t} = x - \frac{4(tx)^3 + 9(tx)^2}{36} + \int_0^x \int_0^t [sy + u^2(s,y)] dy ds, \quad (x,t) \in [0,1]^2$$

with initial condition

$$u(x,0) = 1, 0 \leq x \leq 1$$

and exact solution

$$u(x,t) = xt.$$

Solution Table 2 presents the numerical results for the approximate solution, exact solution and error for different values of x and t between 0 and 1. Figure 4, Figure 5 and Figure 6 give the comparative analysis of the results through visualization.

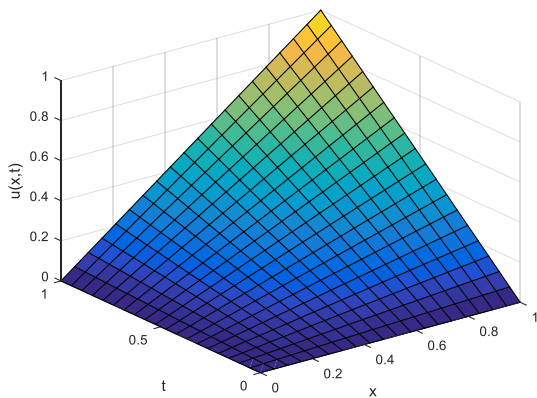


Figure 4: Numerical solution for example 2

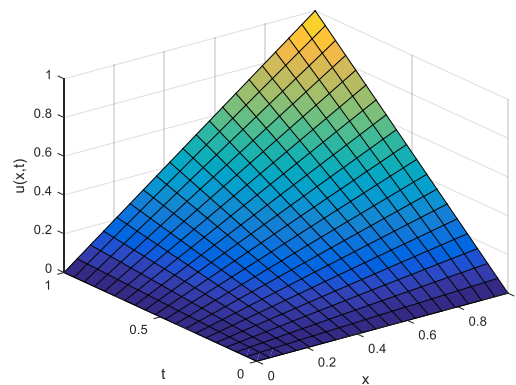


Figure 5: Exact solution for example 2

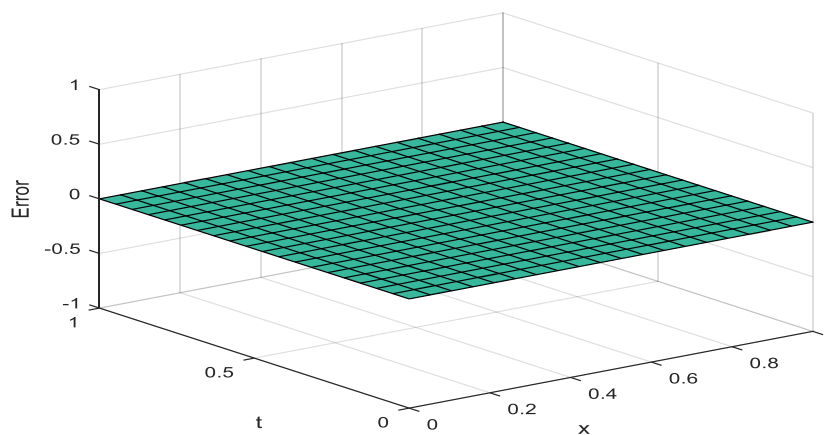


Figure 6: Absolute error for example 2

Table 2: Numerical values at selected points

x	t	Approximate solution	Hussain <i>et al.</i> , (2020)	Exact solution	Error
0.00	0.00	0.00000	0.000000e+00	0.000000e+00	0
0.00	0.25	0.00000	0.000000e+00	0.000000e+00	0
0.00	0.50	0.00000	0.000000e+00	0.000000e+00	0
0.00	0.75	0.00000	0.000000e+00	0.000000e+00	0
0.00	1.00	0.00000	0.000000e+00	0.000000e+00	0
0.25	0.00	0.00000	0.000000e+00	0.000000e+00	0
0.25	0.25	6.25e-02	6.250000e-02	6.250000e-02	0
0.25	0.50	1.25e-01	1.250000e-01	1.250000e-01	0
0.25	0.75	1.88e-01	1.880000e-01	1.880000e-01	0
0.25	1.00	2.50e-01	2.500000e-01	2.500000e-01	0
0.50	0.00	0.00000	0.000000e+00	0.000000e+00	0
0.50	0.25	1.25e-01	1.250000e-01	1.250000e-01	0
0.50	0.50	2.50e-01	2.500000e-01	2.500000e-01	0
0.50	0.75	3.75e-01	3.750000e-01	3.750000e-01	0
0.50	1.00	5.00e-01	5.000000e-01	5.000000e-01	0
0.75	0.00	0.00000	0.000000e+00	0.000000e+00	0
0.75	0.25	1.88 e-01	1.88e-01	1.88e-01	0
0.75	0.50	3.75000	3.750000e-01	3.750000e-01	0
0.75	0.75	5.62500	5.625000e-01	5.625000e-01	0
0.75	1.00	7.50 e-01	7.500000e-01	7.500000e-01	0
1.00	0.00	0.00000	0.000000e+00	0.000000e+00	0
1.00	0.25	2.50e-01	2.500000e-01	2.500000e-01	0
1.00	0.50	5.00e-01	5.000000e-01	5.000000e-01	0
1.00	0.75	7.50e-01	7.500000e-01	7.500000e-01	0
1.00	1.00	1.00e+00	1.000000e+00	1.000000e+00	0

4.2 Discussion

The Jordan block polynomial collocation method is evaluated for accuracy, computational efficiency, and convergence through two test problems (linear and nonlinear partial VIDE). The method achieved excellent accuracy, coinciding with analytical solutions at all tested grid points, producing zero absolute error, due to the polynomial nature of the solutions. Its computational efficiency is marked by immediate convergence and negligible CPU time, effectively reducing costs compared to traditional methods. Convergence behaviour is stable, with increased polynomial degrees leading to spectral-type convergence without instability. Overall, the method is accurate, efficient, and strongly convergent, making it a promising tool in solving both linear and nonlinear partial VIDEs, including potential extension to non-polynomial cases.

A time-dependent partial VIDE is solved numerically utilizing the discretization techniques in the MATLAB code developed. It provides error analysis and computes both exact and numerical solutions. The numerical experiments presented in this work confirm the theoretical results. In both the linear and nonlinear

examples considered, the exact solutions are polynomials, and the Jordan block collocation method recovers them exactly. This validates the uniqueness, convergence, and spectral-type accuracy of the technique.

5.0 CONCLUSION

Both linear and nonlinear two-dimensional partial VIDEs are investigated using collocation technique, and the convergence of this approach is also examined. Without any restricted transformation or presumptions that might alter the problem's physical behavior, the collocation approach produces convergent results that are quick, sequential, and approximate. The collocation solution process is simple, practical, and efficient. Its efficiency is demonstrated on some problems with comparison with the analytical solution.

REFERENCES

- Adamu, S., Aduroja, O. O., Bukar, H. & Onanaye, A. S. (2025). Numerical Solution of 2D Partial Volterra Integro Differential Equations using Polynomial Collocation Matrix Formulation. *FUDMA Journal of Sciences (FJS)*, 9(9), 366-371. DOI:

- <https://doi.org/10.33003/fjs-2025-0909-3851>.
- Adamu, S., Aduroja, O. O. & Bitrus, K. (2023). Numerical Solution to Optimal Control Problems using Collocations Method via Pontrygin's Principle. *FUDMA Journal of Sciences*, 7(5), 228-233. DOI: <https://doi.org/10.33003/fjs-2023-0705-2016>.
- Adesanya, A. O., Osilagun, J. A., Aduroja, O. O. and Adamu, S. (2023). On Approximate Solution of High-Order Linear Fredholm Integro-Differential-Difference Equations with Variable Coefficients using Legendre Collocation Method. *Electronic Journal of Mathematical Analysis and Application (EJMAA)*, 11(1), 198-205. <http://ejmaa.journals.ekb.eg>.
- Aziz, I. & Khan, I. (2017). Numerical Solution of Partial Integro-differential Equations of Diffusion Type. *Math. Probl. Eng.*, 2017, 1-11.
- Burcu G. & Mehmet S. (2017). Numerical Solution of Parabolic-Type Volterra Partial Integro-differential Equations by Laguerre Collocation Method. *International Journal of Applied Physics and Mathematics*, 7(1), 49-58.
- Chen, X (2017). Collocation Methods for Nonlinear Parabolic Partial Differential Equations. A M.Sc. Thesis, Concordia University, Montreal, Quebec, Canada.
- Eashel, A. A., Pishbin, S. & Darania, P. (2025). Convergence Analysis of Multi-Step Collocation Method to first order Volterra Integro Differential Equation with Non-Vanishing Delay. *European Journal of Pure and Applied Mathematics*. 18(2) 5766. DOI: <https://doi.org/10.29020/nybg.ejpam.v18i2.5766>.
- Hussain, A. K., Fadhel, F. S., Yahya, Z. R. & Rusli, N. (2016). Variational Iteration Method (VIM) for solving partial integro differential equations. *Journal of Theoretical and Applied Information Technology*, 88(2), 367-374.
- Hussain, A. K., Fadhel, F. S., Rusli, N. & Yahya, Z. (2020). Iteration Variational Method for solving Two-Dimensional Partial Integro Differential Equations. *J. Phys.: Conf. Ser.* 1591, 1-11.
- Khajehnasiri, A. A. (2015). Numerical Solution of Nonlinear 2D Volterra-Fredholm Integro-Differential Equations by Two-Dimensional Triangular Function. *Int. J. Appl. Comput. Math.*
- Mahdy, A. M. S., Abdou, M. A. & Mohammed D. S. (2023). Computational Methods for solving Higher Order (1+1) Dimensional Mixed Difference Integro-Differential Equations with Variable Coefficients. *Mathematics*, 11, 2045. <https://doi.org/10.3390/math11092045>.
- Micula, S. (2019). On some Iterative Numerical Methods for Mixed Volterra Fredholm Integral Equations. *Symmetry*, 11, 1-10.
- Noori, S. R. M. & Taghizadeh, N. (2020). Study on Solving Two-dimensional Linear and Nonlinear Volterra Partial Integro-differential Equations by Reduced Differential Transform Method. *Applications and Applied Mathematics: An International Journal*, 15(1), 394 - 407.
- Osilagun, J. A., Adesanya, A. O., Anake, T. A. & Adamu, S. (2023). Polynomial Collocation Method for Initial Value Problem of Mixed Integro-Differential Equations. *Mathematics and Computational Sciences (MCS)*, 4(2), 1-12. DOI: <https://doi.org/10.30511/mcs.2023.1972559.1095>.
- Otaide, I. J & Oluwayemi, M. O. (2024). Numerical Treatment of Linear Volterra Integro Differential Equations using Variational Iteration Algorithm with Collocation. *Partial Differential Equations in Applied Mathematics*, 10(2024) 100693. <https://doi.org/10.1016/j.padiff.2024.100693>.
- Pachpatte, B. G. (2011). *Multidimensional Integral Equations and Inequalities*. Atlantis press, Shri Niketen Coloney, Aurangabad, India.
- Sameeh, M. & Elsaid, A. (2016). Chebyshev Collocation Method for Parabolic Partial Integro-differential Equations. *Adv. Math. Phys.*, 2016, 1-7.
- Shalangwa, A. A., Odekunle, M. R. Ade, S. O. & Micheal, C. (2025). Numerical Solution of Two Dimensional Mixed Volterra-Fredholm Integral Equations using Polynomial Collocation Method. *Bima Journal of Sciences and Technology*, 9(1A), 148-156. DOI: <https://doi.org/10.56892/bima.v9i1A.1234>.
- Shalangwa, A. A., Odekunle, M. R. & Ade, S. O. (2025). Numerical Solution of 2D Nonlinear Volterra-Fredholm Integral Equations using Polynomial Collocation Method. *International Journal of Development Mathematics*, 2(1), 22-31. DOI: <https://doi.org/10.62054/ijdm/0201.D2>.
- Tedjani, A. H., Alhazmi, S. E. & Ezz-Eldien, S. S. (2025). An Operational approach for one and two dimension high order multi-pantograph Volterra Integro Differential Equations. *AIMS Mathematics*, 10(4), 9274-9294.
- Zhao, Y. & Zhao, F. (2016). The Analytical Solution of Parabolic Volterra Integro-Differential Equations in the Infinite Domain. *Entropy*, 18, 1-14.