



**A Comparative Study of Two Food Chain Models with and Without Adaptive Traits**

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**Abstract**

*A three – species food chain model consisting of adaptive traits and another one without adaptive traits are presented. We analyzed the existence and the stability of their equilibria and found that the model with adaptive traits is more stable than the model without adaptive traits and also,  $E_0$  in both models are unstable while the states where the three populations persist is locally asymptotically stable under prescribe conditions. Furthermore, from the numerical experiments using hypothetical data, the study confirms the role of adaptive traits as important attributes of the prey for coexistence of species in a three species food chain involving a resource, a prey and a predator.*

**Keywords:** *Adaptive traits; Three- species food chain; Prey; Predator; Resources.*

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**1.0 INTRODUCTION**

In ecosystems, species are connected through intricate trophic relationships defining complex networks, the so – called food chains (Chengjun and Michel, 2009). Understanding the structure and mechanisms underlying the formation of these complex chains is of great importance in ecology. In particular, the food chain structure provides insight into the behavior of ecosystems under perturbations such as the introduction of new species or the extinction of existing species. The classical ecological models of interacting populations typically have focused on two species. They have been applied to plant and animal systems in aquatic and terrestrial environments and across many taxa. However, it has long been recognized that the limited

caricature of ecological systems by two interacting species can account for only a small number of the phenomena that are commonly exhibited in nature (Upadhyay *et al*, 2010). This is particularly true in community studies where the essence of the behavior of a complex system may only be understood when the interactions among a large number of species are incorporated.

Therefore, important behavior that may be critical to community functions may arise only through the interactions of three or more species. Mathematical developments also suggest that community models involving two species as the basic building blocks may miss important ecological behavior. For example, results that are much more complicated than those seen in

two – species models appeared in early theoretical studies of three – species or three trophic – level, models based on local stability analysis (Rosenzweig, 1973). This analysis

**2.0 METHODOLOGY**

**2.1 The Three Species Food Chain Model without Adaptive Traits**

The model is, 
$$\frac{dx}{dt} = x \left[ r \left( 1 - \frac{x}{K} \right) - ay \right],$$
 (2.1.1)

$$\frac{dy}{dt} = y(c_1ax - bz - d_1),$$
 (2.1.2)

$$\frac{dz}{dt} = z(c_2by - d_2),$$
 (2.1.3)

Where the positive parameters,  $r, k, a, b, c_1, c_2, d_1$  and  $d_2$  are defined as follows:

$r$  = the intrinsic rate of natural increase of the resources in the absence of consumers

$k$  = resources carrying capacity,  $a$  = the consumption rate of the prey on the resources

$b$  = predation rate of the predator on the prey

$c_1$  = the conversion efficiency of consumed resources into new prey

$c_2$  = the conversion efficiency of consumed prey into new predators

$d_1$  = the natural death rate of the prey,  $d_2$  = the natural death rate of the predator

$x(t)$  = the resource population at time  $t$ ,  $y(t)$  = the prey population at time  $t$

$z(t)$  = the predator population at time  $t$ .

Model equations (2.1.1) - (2.1.3) are based on the traditional assumption that all species interact with each other via changes in population density. Such interactions are called density mediated interactions.

**2.2 The Three Species Food Chain Model with Adaptive Traits**

Biologists have shown theoretically and empirically that in natural systems, preys often display anti – predator responses (Chengjun and Michel, 2009). That is they alter their foraging behavior on the resources to reduce predation risk. For example seeking refuge or becoming vigilant at the expense of feeding. Altered foraging can result in an increased risk of starvation and hence in reduction in prey density.

Three species food chain model with adaptive traits by Chengjun and Michel (2009) is;

considered only linear approximation to the non – linear equations that ecologists conventionally assume apply to these more complex situations.

$$\frac{dx}{dt} = x \left[ r \left( 1 - \frac{x}{K} \right) - ay \right],$$
 (2.1.4)

$$\frac{dy}{dt} = y(c_1ax - \beta a^2z - d_1),$$
 (2.1.5)

$$\frac{dz}{dt} = z(c_2\beta a^2y - d_2),$$
 (2.1.6)

$$\frac{da}{dt} = \xi(c_1x - 2\beta az),$$
 (2.1.7)

Based on the biological information provided above, it is assumed that the two consumption rates  $a$  and  $b$  are adaptive traits in the model (2.1.4) – (2.1.7). These two traits are positively related to each other, such that prey increases resources consumption (greater  $a$ ) at the expense of a higher predation risk by carnivores (greater  $b$ ), which implies that herbivores can be caught more easily by carnivores when they are actively searching for and consuming food. Here, it is assumed that  $b = \beta a^2 (\beta > 0)$

Where the constant  $\beta$  measures the predation risk incurred by an individual prey per unit foraging time per unit density of the predator. The parameter  $\xi$  scales the rate at which the adaptive traits change.

**Theorem 2.1 Stability Criteria**

Suppose that the  $n \times n$  matrix  $A$  has eigenvalues  $\lambda_i, i = 1, 2, \dots, n$ . Then the stability of any solution of

$$x' = Ax$$

is determined according to the following criteria:

- i. If  $Re(\lambda_i) < 0$ , for all  $i$  then there is a uniform and asymptotic stability.
- ii. If  $Re(\lambda_i) \leq 0$  for all  $i$  and the algebraic multiplicity equals the geometric multiplicity whenever  $\lambda_i = 0$  for any  $i$ , then there is uniform stability.
- iii. If  $Re(\lambda_i) > 0$  for at least one  $i$  or the algebraic multiplicity is greater than the geometric multiplicity should  $Re(\lambda_i) = 0$  for any  $i$ , then there is instability.

### 3.0 EQUILIBRIUM AND STABILITY STATES

#### 3.1 Equilibrium States of the Model without Adaptive Traits

Here, we find the equilibrium states of the model without adaptive traits.

Let  $m = \frac{r}{K}$ , then from (2.1.1) – (2.1.3) and at equilibrium, we have

$$x[m(k - x) - ay] = 0 \tag{3.1.1}$$

$$y(c_1ax - bz - d_1) = 0 \tag{3.1.2}$$

$$z(c_2by - d_2) = 0 \tag{3.1.3}$$

From (3.1.1) - (3.1.3), it is clear that  $E_0 = (0, 0, 0)$ , - the trivial equilibrium state of the model,

$$E_1 = (K, 0, 0) \quad E_2 = \left( \frac{d_1}{c_1a}, \frac{m(ac_1K - d_1)}{c_1a^2}, 0 \right), \text{ provided that } K > \frac{d_1}{ac_1} \text{ and if}$$

$$a > \frac{a^2c_1d_2}{c_2m(ac_1K - d_1)}, K > \frac{d_1}{ac_1}, E_3 = \left( \frac{bc_2Km - ad_2}{mc_2b}, \frac{d_2}{c_2b}, \frac{c_1a(c_2bKm - ad_2) - bc_2d_1m}{mc_2b^2} \right)$$

#### 3.2 Stability of the Equilibrium States of the Model Without Adaptive Traits.

Let  $f = x[m(K - x) - ay]$ ,  $g = y(c_1ax - bz - d_1)$ ,  $h = z(c_2by - d_2)$ , then

$$\begin{aligned} \frac{df}{dx} &= m(K - 2x) - ay, & \frac{df}{dy} &= -ax, & \frac{df}{dz} &= 0 \\ \frac{dg}{dx} &= c_1ay, & \frac{dg}{dy} &= c_1ax - bz - d_1, & \frac{dg}{dz} &= -by \end{aligned}$$

$$\frac{dh}{dx} = 0, \quad \frac{dh}{dy} = c_2bz, \quad \frac{dh}{dz} = c_2by - d_2$$

**Theorem 3.1.1:**  $E_0$  exists but is unstable.

**Proof:**

The Jacobian matrix is given by

$$J_{E_0} = \begin{pmatrix} mK & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

Therefore, the characteristic polynomial of  $J_{E_0}$  is

$$P(\lambda) = \begin{vmatrix} mK - \lambda & 0 & 0 \\ 0 & -d_1 - \lambda & 0 \\ 0 & 0 & -d_2 - \lambda \end{vmatrix} = (mK - \lambda)(-d_1 - \lambda)(-d_2 - \lambda)$$

Therefore, the characteristics equation is

$$(mK - \lambda)(-d_1 - \lambda)(-d_2 - \lambda) = 0 \tag{3.1.4}$$

Solving (3.1.4) gives

$$\therefore \lambda_1 = mK, \lambda_2 = -d_1, \lambda_3 = -d_2$$

Since  $\lambda_1 > 0$ , by Theorem 3.1(iii),  $E_0 = (0,0,0)$  is an unstable equilibrium point.

**Theorem 3.1.2:**  $E_1$  exists and is unstable if  $K > \frac{d_1}{ac_1}$ .

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**Proof:**

The Jacobian matrix is given by

$$J_{E_1} = \begin{pmatrix} -mK & -aK & 0 \\ 0 & ac_1K - d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

Therefore, the characteristic polynomial of  $J_{E_1}$  is

$$P(\lambda) = \begin{vmatrix} -mK - \lambda & 0 & 0 \\ 0 & ac_1K - d_1 - \lambda & 0 \\ 0 & 0 & -d_2 - \lambda \end{vmatrix} \\ = (-mK - \lambda)(ac_1K - d_1 - \lambda)(-d_2 - \lambda)$$

Therefore, the characteristics equation is

$$(-mK - \lambda)(ac_1K - d_1 - \lambda)(-d_2 - \lambda) = 0 \quad (3.1.5)$$

Solving (3.1.5) gives

$$\therefore \lambda_1 = -mK, \lambda_2 = ac_1K - d_1 > 0, \lambda_3 = -d_2$$

Since  $\lambda_2 > 0$ , by hypothesis, it follows from Theorem 3.1.1(iii) that  $E_1$  is an unstable equilibrium point.

**Theorem 3.1.3:** If  $K > \frac{d_1}{ac_1} + \frac{ad_2}{bc_2m'}$ , then  $E_2$  exists and is unstable.

**Proof:**

For  $E_2 = \left( \frac{d_1}{c_1a}, \frac{m(ac_1K - d_1)}{c_1a^2}, 0 \right)$ , the Jacobian matrix is given by

$$J_{E_2} = \begin{pmatrix} -\frac{d_1m}{ac_1} & \frac{-d_1}{c_1} & 0 \\ \frac{m(ac_1K - d_1)}{a} & 0 & \frac{-bm(ac_1K - d_1)}{c_1a^2} \\ 0 & 0 & \frac{bc_2m(ac_1K - d_1)}{c_1a^2} - d_2 \end{pmatrix}$$

$$\text{Let } A_{11} = -\frac{d_1m}{ac_1}, A_{12} = \frac{-d_1}{c_1}, A_{21} = \frac{m(ac_1K - d_1)}{a}, A_{23} = \frac{-bm(ac_1K - d_1)}{c_1a^2}, A_{33} = \frac{bc_2m(ac_1K - d_1)}{c_1a^2} - d_2$$

Hence, we have the characteristic polynomial as

$$P(\lambda) = \begin{vmatrix} A_{11} - \lambda & A_{12} & 0 \\ A_{21} & -\lambda & A_{23} \\ 0 & 0 & A_{33} - \lambda \end{vmatrix} = (A_{33} - \lambda)(\lambda^2 + \lambda A + B)$$

Therefore, the characteristic equation for  $J_{E_2}$  is

$$(A_{33} - \lambda)(\lambda^2 + \lambda A + B) = 0 \quad (3.1.6)$$

Where  $A = -(A_{11})$  and  $B = -A_{12}A_{21}$ .

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Therefore,

$$\begin{aligned} \lambda_1 = A_{33} &= \frac{bc_2m(ac_1k - d_1)}{c_1a^2} - d_2 = \frac{bc_2m(ac_1k - d_1) - a^2c_1d_2}{a^2c_1} \\ &= \frac{abc_1c_2Km - bc_2d_1m - a^2c_1d_2}{a^2c_1} = \frac{abc_1c_2Km}{a^2c_1} - \frac{bc_2d_1m}{a^2c_1} - \frac{a^2c_1d_2}{a^2c_1} \\ &= K - \left(\frac{d_1}{ac_1} + \frac{ad_2}{bc_2m}\right) > 0 \text{ for } K > \frac{d_1}{ac_1} + \frac{ad_2}{bc_2m} \text{ and} \end{aligned}$$

$$\lambda_{2,3} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}, \text{ where } A = -(A_{11}) \text{ and } B = -A_{12}A_{21}$$

Since  $\lambda_1 > 0$ , we conclude that  $E_2$  is unstable (Theorem 3.1.1)

**Theorem 3.1.4:**  $E_3$  exists and is locally asymptotically stable If  $P_{11} < 0, P_{22} < 0$  and  $P_{33} < 0$ .

**Proof:**

The Jacobian matrix is,  $J_{E_3} = \begin{pmatrix} m(k - 2\bar{x}) - a\bar{y} & -a\bar{x} & 0 \\ c_1 a\bar{y} & c_1 a\bar{x} - b\bar{z} - d_1 & -b\bar{y} \\ 0 & c_2 b\bar{z} & c_2 b\bar{y} - d_2 \end{pmatrix}$

Therefore, the characteristic polynomial of  $J_{E_3}$  is given by

$$P(\lambda) = \begin{vmatrix} m(k - 2\bar{x}) - a\bar{y} - \lambda & -a\bar{x} & 0 \\ c_1 a\bar{y} & c_1 a\bar{x} - b\bar{z} - d_1 - \lambda & -b\bar{y} \\ 0 & c_2 b\bar{z} & c_2 b\bar{y} - d_2 - \lambda \end{vmatrix}$$

Let;

$$\begin{aligned} P_{11} &= m(k - 2\bar{x}) - a\bar{y} & P_{12} &= -a\bar{x} < 0 & P_{13} &= 0 \\ P_{21} &= c_1 a\bar{y} & P_{22} &= c_1 a\bar{x} - b\bar{z} - d_1 & P_{23} &= -b\bar{y} < 0 \\ P_{31} &= 0 & P_{32} &= c_2 b\bar{z} & P_{33} &= c_2 b\bar{y} - d_2 \end{aligned}$$

Therefore, the characteristic polynomial becomes;

$$P(\lambda) = \begin{vmatrix} P_{11} - \lambda & P_{12} & 0 \\ P_{21} & P_{22} - \lambda & P_{23} \\ 0 & P_{32} & P_{33} - \lambda \end{vmatrix} = \lambda^3 + \lambda^2 A + \lambda B + C$$

Hence, the characteristic equation of  $J_{E_3}$  is

$$\lambda^3 + \lambda^2 A + \lambda B + C = 0 \tag{3.1.7}$$

Where  $A = -(P_{11} + P_{22} + P_{33}), B = (P_{11}P_{22} + P_{11}P_{33} + P_{22}P_{33} - P_{23}P_{32} - P_{12}P_{21})$

and  $C = (P_{11}P_{23}P_{32} + P_{12}P_{21}P_{33} - P_{11}P_{22}P_{33})$ .

By the Routh – Hurwitz criteria, a set of necessary and sufficient conditions for all the roots of (3.1.7) to have negative real parts are;

1.  $A > 0$ , 2.  $B > 0$ , 3.  $C > 0$ , 4.  $AB - C > 0$  or  $AB - C > 0$ .
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**Proof:**

$$A = -\{-(P_{11} + P_{22} + P_{33})\} = P_{11} + P_{22} + P_{33} > 0$$

$$\begin{aligned} B &= (-P_{11})(-P_{22}) + (-P_{11})(-P_{33}) + (-P_{22})(-P_{33}) - (-P_{23})P_{32} - (-P_{12})P_{21} \\ &= P_{11}P_{22} + P_{11}P_{33} + P_{22}P_{33} + P_{23}P_{32} + P_{12}P_{21} > 0 \end{aligned}$$

$$\begin{aligned} C &= (-P_{11})(-P_{23})P_{32} + (-P_{12})P_{21}(-P_{33}) - (-P_{11})(-P_{22})(-P_{33}) \\ &= P_{11}P_{23}P_{32} + P_{12}P_{21}P_{33} + P_{11}P_{22}P_{33} > 0 \end{aligned}$$

$$\begin{aligned} AB - C &= (P_{11} + P_{22} + P_{33})(P_{11}P_{22} + P_{11}P_{33} + P_{22}P_{33} + P_{23}P_{32} + P_{12}P_{21}) \\ &\quad - (P_{11}P_{23}P_{32} + P_{12}P_{21}P_{33} + P_{11}P_{22}P_{33}) \\ &= P_{11}^2(P_{22} + P_{33}) + P_{22}^2(P_{11} + P_{33}) + P_{33}^2(P_{11} + P_{22}) + 2P_{11}P_{22}P_{33} + \\ &\quad P_{11}P_{12}P_{21} + P_{22}P_{23}P_{32} + P_{12}P_{21}P_{22} + P_{23}P_{32}P_{33} > 0 \end{aligned}$$

Hence,  $E_3$  is locally asymptotically stable.

### 3.3 Equilibrium States of the Model with Adaptive Traits

Let  $m = \frac{r}{K}$ , then from (2.1.4) – (2.1.7) and at equilibrium, we have

$$x[m(K - x) - ay] = 0 \tag{3.2.1}$$

$$y(c_1ax - \beta a^2z - d_1) = 0 \tag{3.2.2}$$

$$z(c_2\beta a^2y - d_2) = 0 \tag{3.2.3}$$

$$\xi(c_1x - 2\beta az) = 0 \tag{3.2.4}$$

From (3.2.1) - (3.2.4), we have that  $E_0 = (0, 0, 0, 0)$ , the trivial state and

$E_1 = \left( \frac{2\beta c_2 d_1 K m}{2\beta c_2 d_1 m + c_1 d_2}, \frac{c_1^2 c_2 \beta d_2 K^2 m^2}{(2\beta c_2 d_1 m + c_1 d_2)^2}, \frac{\beta c_1 c_2^2 d_1 K^2 m^2}{(2\beta c_2 d_1 m + c_1 d_2)^2}, \frac{2\beta c_2 d_1 m + c_1 d_2}{c_1 c_2 \beta K m} \right)$  is the other equilibrium state (where three populations persist)

### 3.4 Stability of the Equilibrium States of the Model with Adaptive Traits

Now, let  $f = x[m(K - x) - ay]$ ,  $g = y(c_1ax - \beta a^2z - d_1)$ ,  $h = z(c_2\beta a^2y - d_2)$  and  $i = \xi(c_1x - 2\beta az)$ , then

$$\frac{\partial f}{\partial x} = m(K - 2x) - ay, \quad \frac{\partial f}{\partial y} = -ax, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial a} = -xy$$

$$\frac{\partial g}{\partial x} = c_1ay, \quad \frac{\partial g}{\partial y} = c_1ax - \beta a^2z - d_1, \quad \frac{\partial g}{\partial z} = -\beta a^2y, \quad \frac{\partial g}{\partial a} = y(c_1x - 2\beta az).$$


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$$\frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial y} = \beta c_2 a^2 z, \quad \frac{\partial h}{\partial z} = c_2 \beta a^2 y - d_2, \quad \frac{\partial h}{\partial a} = 2\beta c_2 a y z$$

$$\frac{\partial i}{\partial x} = \xi c_1, \quad \frac{\partial i}{\partial y} = 0, \quad \frac{\partial i}{\partial z} = -2\xi\beta a, \quad \frac{\partial i}{\partial a} = -2\xi\beta z$$

**Theorem 3.3.1:**  $E_0$  exists but is unstable.

**Proof:**

The Jacobian matrix is

$$J_{E_0} = \begin{pmatrix} mK & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ \xi c_1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial corresponding to the above matrix is

$$P(\lambda) = \begin{vmatrix} mK - \lambda & 0 & 0 & 0 \\ 0 & -d_1 - \lambda & 0 & 0 \\ 0 & 0 & -d_2 - \lambda & 0 \\ \xi c_1 & 0 & 0 & -\lambda \end{vmatrix} = \lambda(mK - \lambda)(-d_1 - \lambda)(d_2 + \lambda)$$

Therefore, the characteristics equation is

$$\lambda(mK - \lambda)(-d_1 - \lambda)(d_2 + \lambda) = 0 \quad (3.3.1)$$

Solving (3.3.1) gives

$$\lambda_1 = 0, \lambda_2 = mK > 0, \lambda_3 = -d_1, \lambda_4 = -d_2$$

Since  $\lambda_2 > 0$ , by Theorem 3.1.1(iii),  $E_0$  is an unstable equilibrium point.

**Theorem 3.3.2:**  $E_1$  exists and is locally asymptotically stable.

**Proof:**

The Jacobian matrix is

$$J_{E_1} = \begin{pmatrix} r(1 - 2\bar{x}) - \bar{a}\bar{y} & -\bar{a}\bar{x} & 0 & -\bar{x}\bar{y} \\ m\bar{a}\bar{y} & m\bar{a}(\bar{x} - \bar{a}\bar{z}) - d_1 & -m\bar{a}^2\bar{y}\bar{y}(m\bar{x} - 2m\bar{a}\bar{z}) & \\ 0 & n\bar{a}^2\bar{z} & n\bar{a}^2\bar{y} - d_2 & 2n\bar{a}\bar{y}\bar{z} \\ \xi & 0 & -2\xi\bar{a} & -2\xi\bar{z} \end{pmatrix}$$

Let

$$\begin{aligned} B_{11} &= r(1 - 2x^*) - a^*y^* = r \left( 1 - 2 \times \frac{2d_1nr}{2d_1nr + d_2m} \right) - \frac{2d_1nr + d_2m}{mnr} \times \frac{d_2m^2nr^2}{(2d_1nr + d_2m)^2} \\ &= r - \frac{4d_1nr}{2d_1nr + d_2m} - \frac{d_2mr}{2d_1nr + d_2m} \end{aligned}$$


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$$= \frac{2d_1nr^2 - 4d_1nr^2}{2d_1nr + d_2m} = -\frac{2d_1nr^2}{2d_1nr + d_2m}$$

Similarly,

$$B_{12} = -\frac{d_1nr}{2d_1nr + d_2m} B_{14} = -\frac{2d_1d_2m^2n^2r^3}{(2d_1nr + d_2m)^3} B_{21} = \frac{d_2m^2r}{2d_1nr + d_2m} B_{22} = 0 \quad B_{23} = -\frac{md_2}{n}$$

$$B_{24} = 0, \quad B_{32} = \frac{d_1n}{m} B_{33} = 0, \quad B_{34} = \frac{2d_1d_2m^2n^3r^3}{(2d_1nr + d_2m)^3}, \quad B_{43} = -\frac{2\xi(2d_1nr + d_2m)}{mnr}$$

$$B_{44} = -\frac{2\xi d_1mn^2r^2}{(2d_1nr + d_2m)^2}$$

Hence, the characteristic polynomial of  $J_{E_2}$  is

$$P(\lambda) = \begin{vmatrix} B_{11} - \lambda & B_{12} & 0 & B_{14} \\ B_{21} & B_{22} - \lambda & B_{23} & B_{24} \\ 0 & B_{32} & B_{33} - \lambda & B_{34} \\ \xi & 0 & B_{43} & B_{44} - \lambda \end{vmatrix} = \lambda^4 + \lambda^3A + \lambda^2B + \lambda C + D$$

Therefore, the characteristic equation is;  $\lambda^4 + \lambda^3A + \lambda^2B + \lambda C + D = 0$

Where,  $A = (-B_{11} - B_{22} - B_{33} - B_{44})$

$$B = B_{11}B_{22} + B_{11}B_{33} + B_{11}B_{44} + B_{22}B_{33} + B_{22}B_{44} + B_{33}B_{44} - B_{34}B_{43} - B_{24}B_{32} - B_{12}B_{21} + \xi B_{24}$$

$$C = B_{11}B_{34}B_{43} + B_{11}B_{23}B_{32} + B_{22}B_{34}B_{43} + B_{23}B_{32}B_{44} + B_{12}B_{21}B_{33} + B_{12}B_{24}B_{44} + \xi B_{12}B_{24}$$

$$-B_{11}B_{22}B_{33} - B_{11}B_{22}B_{44} - B_{11}B_{33}B_{44} - B_{22}B_{33}B_{44} - B_{24}B_{32}B_{43} - \xi B_{14}B_{22} - \xi B_{14}B_{33}$$

$$D = B_{11}B_{22}B_{33}B_{44} + B_{11}B_{24}B_{32}B_{43} + B_{12}B_{24}B_{43}B_{34} + \xi B_{12}B_{24}B_{34} + \xi B_{14}B_{22}B_{33}$$

$$-B_{11}B_{22}B_{34}B_{43} - B_{12}B_{21}B_{33}B_{44} - B_{11}B_{24}B_{32}B_{44} - B_{14}B_{21}B_{32}B_{43} - \xi B_{12}B_{24}B_{33}$$

$$-\xi^2 B_{14}B_{22}B_{23}$$

By the Routh – Hurwitz criteria, the necessary and sufficient conditions for the roots of  $P(\lambda)$  to have negative real parts are;

1.  $A > 0$ , 2.  $B > 0$ , 3.  $C > 0$ , 4.  $D > 0$ , 5.  $ABC - A^2D > C^2$  or  $ABC - A^2D - C^2 > 0$

**Proof:**

Since  $B_{22} = 0, B_{33} = 0$  and  $B_{24} = 0$

$$A = (-B_{11} - B_{22} - B_{33} - B_{44}) = -(B_{11} + B_{44}) = -\left(-\frac{2d_1nr^2}{2d_1nr + d_2m} - \frac{2\xi d_1mn^2r^2}{(2d_1nr + d_2m)^2}\right)$$


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$$= \frac{2d_1nr^2}{2d_1nr + d_2m} + \frac{2\xi d_1mn^2r^2}{(2d_1nr + d_2m)^2} > 0$$

Similarly,

$$B = B_{11}B_{44} - B_{34}B_{43} - B_{12}B_{21} = \frac{d_1d_2mnr(4n\xi + m)}{(2d_1nr + d_2m)^2} + \frac{2d_1^2mn^3r^4\xi}{(2d_1nr + d_2m)^3} > 0$$

$$C = B_{11}B_{34}B_{43} + B_{11}B_{23}B_{32} + B_{23}B_{32}B_{44} = \frac{2d_1nr^2}{2d_1nr + d_2m} + \frac{2d_1^2d_2mn^2r^2\xi}{(2d_1nr + d_2m)^2} + \frac{8d_1^2d_2mn^3r^4\xi}{(2d_1nr + d_2m)^3}$$

$$D = -B_{14}B_{21}B_{32}B_{43} = \frac{4d_1^3d_2mn^3r^4\xi}{(2d_1nr + d_2m)^3} > 0$$

Next, we show that  $ABC - A^2D - C^2 > 0$ .

$ABC - A^2D - C^2 > 0$  is equivalent to

$$16d_1^4d_2n^4r^4 + 8d_1^3d_2mn^3r^3 + 4d_1^2d_2mn^2r^2[6md_2^2 + 3mnd_2\xi + 2nr\xi(m + 2nr + n^2\xi)] \\ + 2d_1mnr[4d_2^4m^2 + 3d_2^3m^2n\xi + 2d_2mn^4r\xi^3 + 2n^3r^3\xi^2(m + 2nr) \\ + 4d_2^2mnr\xi(m + 2nr + n^2\xi)] + m^2[d_2^5m^2 + d_2^4m^2n\xi + 2d_2^2mn^4r\xi^3 + 2n^4r^3\xi^3(m + 2nr) \\ + 2d_2n^3r^2\xi^2(m + 2nr)(r + 2n\xi) + 2d_2^3mnr\xi(m + 2nr + n^2\xi)] > 0$$

Therefore,  $E_1$  is locally asymptotically stable.

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## 4.0 DISCUSSION OF FINDINGS/RESULTS

### 4.1 Stability Analysis

The results from stability analysis of the models are presented below.

#### 4.1.1 The Model Without Adaptive Traits

- a.  $E_0$  exists but is unstable. This implies that little perturbation in the system may cause the state to move to any of the other states.
- b.  $E_1$  exists and is unstable if  $K > \frac{d_1}{ac_1}$ . The system at this state is unstable if the carrying capacity,  $K > \frac{d_1}{ac_1}$  otherwise it is stable and prey population grows to the carrying capacity,  $K$ .
- c. If  $K > \frac{d_1}{ac_1} + \frac{ad_2}{bc_2m}$ , then  $E_2$  exists and is unstable. In the absence of the predator the resource and the prey

population will not persist under the prescribed conditions.

- d.  $E_3$  exists and is locally asymptotically stable If  $P_{11} < 0, P_{22} < 0$  and  $P_{33} < 0$ .

#### 4.1.2 The Model with Adaptive Traits

- a.  $E_0$  exists but is unstable. It implies that little perturbation in the system may cause the state to evolve to the other state.
- b.  $E_1$  exists and is locally asymptotically stable.

### 4.2 Numerical Experiments

We used the fourth – order Runge – Kutta method (RK4) in our numerical analysis. Therefore, hypothetical values are assigned to the parameters as shown in the table below.

Table 4.1: Table of parameter values for the numerical experiments

Figure	1	2
Parameter		
$a$	1	1
$b$	0.2	0.4
$r$	2	2
$k$	10	10
$c_1$	0.5	0.5
$c_2$	0.5	0.5
$d_1$	0.1	0.1
$d_2$	0.1	0.1
$\beta$	0.4	0.6
$\xi$	0.2	0.2
$x(0)$	8	8
$y(0)$	6	6
$z(0)$	2	2
$a(0)$	0.01	0.01

As shown above, the numerical analysis is meant to study the following situations.

1. The situation where the consumption rate of the prey on the resource,  $a$  is 1 and predation rate of the predator on the prey,  $b$  is 0.2.

2. The situation where the consumption rate of the prey on the resource,  $a$  is 1 and predation rate of the predator on the prey,  $b$  is 0.4.

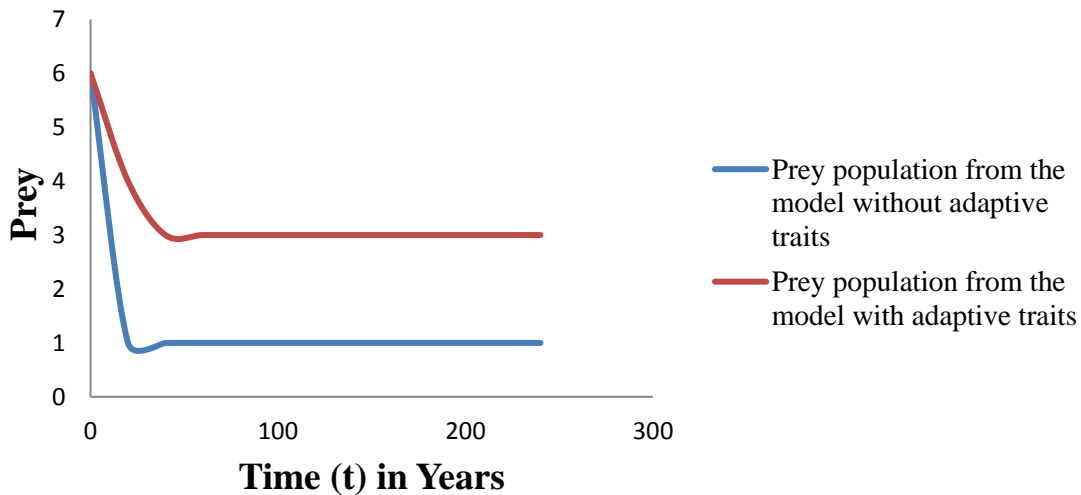


Figure 1. Graph showing the results of simulations from the models involving the prey.

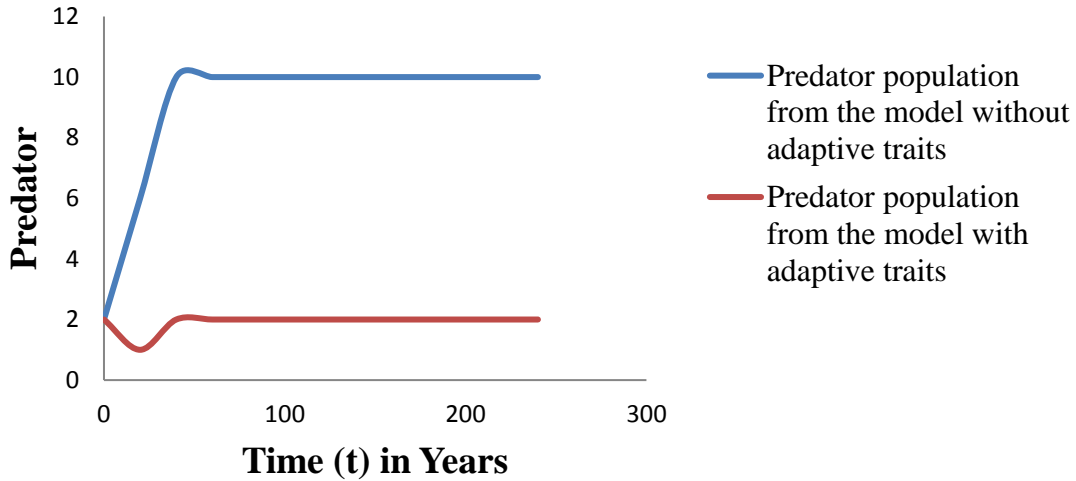


Figure 2. Graph showing the results of simulations from the models involving the predator.

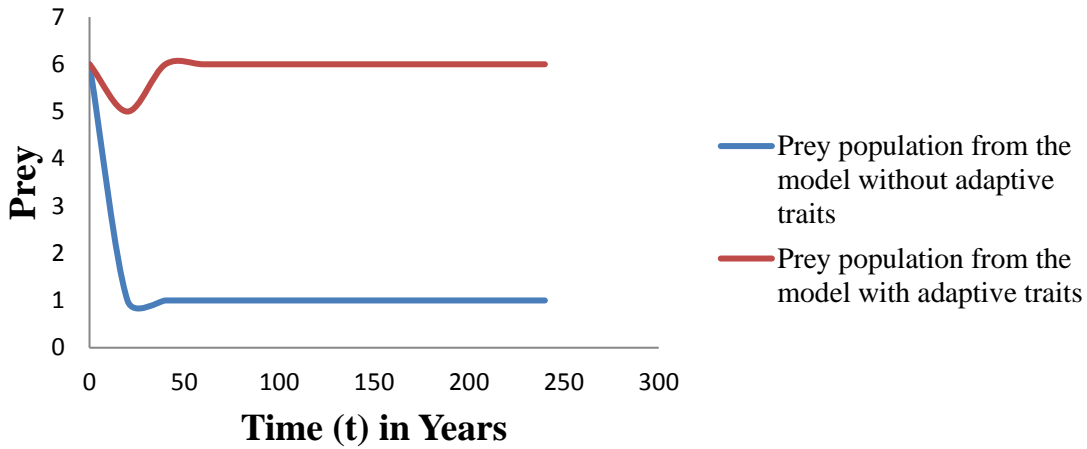


Figure 3. Graph showing the results of simulations from the models involving the prey.

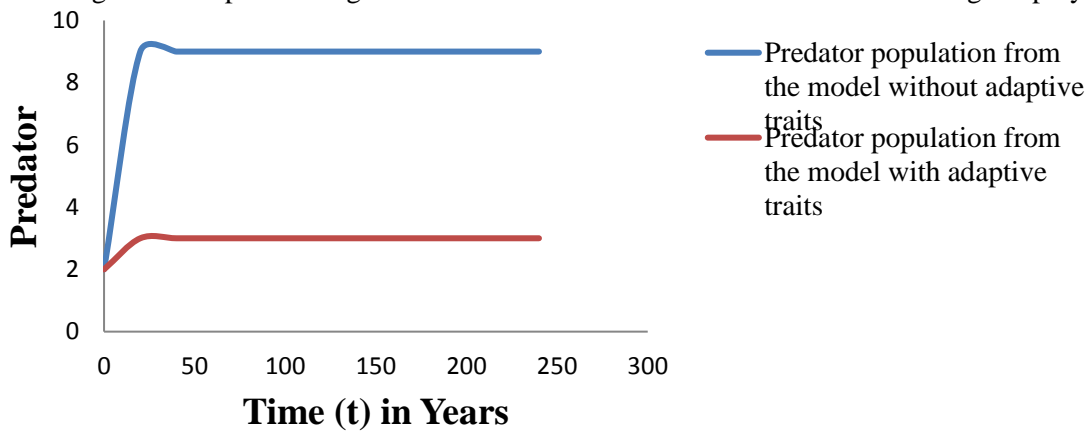


Figure 4. Graph showing the results of simulations from the models involving the predator.

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#### 4.2.1 Experiment One

Here, we study the models when predation rate of the predator on the prey,  $b$  is 0.2. The results from experiment show that:

1. Prey population from the model with adaptive traits is greater than the prey population from the model without adaptive traits (Figure 1).
2. Predator population from the model without adaptive traits is greater than the predator population from the model with adaptive traits (Figure 2).

#### 4.2.2 Experiment Two

In this case, we examined the models when the predation rate of the predator on the prey,  $b$  increases from 0.2 to 0.4. We also observed that:

1. Prey population from the model with adaptive traits is more than the prey population from the model without adaptive traits (Figure 3).
2. Predator population from the model without adaptive traits is more than the predator population from the model with adaptive traits (Figure 4).

#### 5.0 CONCLUSION

The research work, have reviewed and studied a three species food chain model with and without adaptive traits. The model parameters/variables are given in Table 4.1. Runge-kuta method for numerical analysis was used in the numerical experiments.

Existence and stability of the equilibrium states of the two models have been investigated. It was found that, the model with adaptive traits is more stable than that without adaptive traits (the model without adaptive traits has more equilibrium states than the model with adaptive traits). Numerical results from the numerical experiments carried out also indicated that in model with adaptive traits, as the predation rate increases, it has little effects on the prey because of its adaptive traits but has serious consequences on the prey from the model without adaptive traits. Also, predator in the model without adaptive traits is greater because of the absence of adaptive traits in the model.

The study therefore confirms the role of adaptive traits as important attributes of the prey for coexistence of species in a three species food chain involving a resource, a prey and a predator.

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