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A Generalized Family of Ratio-Product Exponential Type Estimators
Using Power Transformation

¹Apantaku, F.S, ¹Olayiwola, O.M, ²Ibharalu, F.T, ¹Alagbe, A. A, ^{1,3}Ishaq, O. O, ¹Olabulo, S. O

¹Department of Statistics, Federal University of Agriculture, Abeokuta, Nigeria.

²Department of Computer, Federal University of Agriculture, Abeokuta, Nigeria.

³Department of Statistics, Aliko Dangote University of Science and Technology, Wudil, Nigeria.

Corresponding Author's Email & Phone No.: adeolaalagbe05@gmail.com; 08070739030

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ABSTRACT

In this research study, a new population mean estimator for two auxiliary variables in stratified random sampling, based on power transformation and exponential techniques, was proposed, following the approach developed by Tailor et al. in 2015. We derived the bias and mean square error (MSE) of this proposed estimator up to the second order of approximation. The conditions in which this newly proposed estimator outperforms the existing ones were obtained. Empirical findings from the study confirm the superior efficiency of the proposed estimator compared to the existing ones.

Keywords: Parameter, Ratio, Exponential, Estimator, Variable

1.0 INTRODUCTION

Estimating population parameters like mean, variance, and ratios is common in fields like agriculture, economics, medicine, and population studies. Researchers have used additional information to enhance estimators regardless of sampling methods. Audu et al., (2017), Muili et al., (2019), Olayiwola et al., (2021), Abiodun et al., (2021) and others used known auxiliary variable parameters for simple random sampling. For two auxiliary variables. Recent work, including Zaman (2020) focused on exponential estimators. Audu et al., (2023) and others generalized existing estimators and introduced exponential-type estimators. Kadilar and Cingi (2003) used the coefficient of variation and kurtosis for a ratio-type estimator. Singh et al. (2013) suggested power transformation estimators. Tailor et al. (2013) discussed dual-to-ratio and product-type exponential estimators. Tailor et al. (2015) defined improved ratio estimators under stratified random

Now we define;

$$\bar{Y}_h = \frac{1}{N_h} \sum_{i=1}^{N_h} y_{hi} : k^{\text{th}} \text{ Stratum mean for the study variable } Y.$$

$$\bar{X}_h = \frac{1}{N_h} \sum_{i=1}^{N_h} x_{hi} : k^{\text{th}} \text{ Stratum mean for the study variable } X.$$

sampling, outperforming traditional methods, particularly in simple random sampling with certain limitations.

1.1 Procedure, Notations, and Definitions

Consider a finite population $U = U_1, U_2, U_3, \dots, U_N$ of size N divided into L strata of size N_h ($h = 1, 2, \dots, L$). Within each stratum, we have three variables: the study variable y , and two auxiliary variables x and z taking values y_{hi}, x_{hi} , and z_{hi} ($h = i, 2, \dots, L; i = 1, 2, \dots, N_i$) respectively, on the i th unit of the h th stratum. x is positively correlated with y , while z is negatively correlated with y . we drew a sample of size n_h from each stratum, forming a total sample of size $n = \sum_{h=1}^L n_h$.

$$\bar{Z} = \frac{1}{N_h} \sum_{i=1}^{N_h} z_{hi} : k^{\text{th}} \text{ Stratum mean for the study variable Z.}$$

$$\bar{Y} = \sum_{h=1}^L W_h \bar{Y}_h : \text{Population mean of the study variable Y.}$$

$$\bar{X} = \frac{1}{N} \sum_{h=1}^L W_h \bar{X} : \text{Population mean of the study variable X.}$$

$$\bar{Z} = \sum_{h=1}^L W_h \bar{Z}_h : \text{Population mean of the study variable Z.}$$

$$\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi} : \text{Sample mean for the study variable Y for } k^{\text{th}} \text{ stratum.}$$

$$\bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{hi} : \text{Sample mean for the study variable X for } k^{\text{th}} \text{ stratum.}$$

$$\bar{z}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} z_{hi} : \text{Sample mean for the study variable Z for } k^{\text{th}} \text{ stratum.}$$

$W_h = \frac{N_h}{N}$ is the weight of h th stratum.

Where;

\bar{y}_{st} , \bar{x}_{st} and \bar{z}_{st} are means of sample taken from h th stratum for variates y , x and z respectively.

$$\bar{y}_{st} = \sum_{h=1}^L W_h \bar{y}_h : \text{Unbiased estimator of the population mean } \bar{Y}$$

$$\bar{x}_{st} = \sum_{h=1}^L W_h \bar{x}_h : \text{Unbiased estimator of the population mean } \bar{X}$$

$$\bar{z}_{st} = \sum_{h=1}^L W_h \bar{z}_h : \text{Unbiased estimator of the population mean } \bar{Z}$$

The variance of unbiased estimator \bar{y}_{st} , \bar{x}_{st} and \bar{z}_{st} are;

$$Var(\bar{y}_{st}) = \sum_{h=1}^L W_h^2 \psi_h S_{yh}^2 \tag{1}$$

$$Var(\bar{x}_{st}) = \sum_{h=1}^L W_h^2 \psi_h S_{xh}^2 \tag{2}$$

$$Var(\bar{z}_{st}) = \sum_{h=1}^L W_h^2 \gamma_h S_{zh}^2 \tag{3}$$

Kadilar and Cingi (2003) defined y_{sidw} , y_{sh} , y_{upsi1} and y_{upsi2} in stratified random sampling respectively as

$$y_{sidw}^{st} = \bar{y}_{st} \left[\frac{W_h (\bar{X}_h + C_{xh})}{\sum_{h=1}^k W_h (\bar{x}_h + C_{xh})} \right] \tag{4}$$

$$y_{sh}^{st} = \bar{y}_{st} \left[\frac{\sum_{h=1}^k W_h (\bar{X}_h + \beta_{2h}(x))}{\sum_{h=1}^k W_h (\bar{x}_h + \beta_{2h}(x))} \right] \tag{5}$$

$$y_{\text{upsi}_1}^{st} = \bar{y}_{st} \left[\frac{\sum_{h=1}^k W_h (\bar{X}_h \beta_{2h}(x) + C_{xh})}{\sum_{h=1}^k W_h (\bar{x}_h \beta_{2h}(x) + C_{xh})} \right] \tag{6}$$

$$y_{\text{upsi}_2}^{st} = \bar{y}_{st} \left[\frac{\sum_{h=1}^k W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))}{\sum_{h=1}^k W_h (\bar{x}_h C_{xh} + \beta_{2h}(x))} \right] \tag{7}$$

To the first degree of approximation mean square errors of y_{sidw} , y_{sh} , y_{upsi_1} and y_{upsi_2} respectively are;

$$MSE(y_{\text{sidw}}^{st}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_{\text{sidw}}^2 S_{xh}^2 - 2R_{\text{sidw}} S_{yhx}) \tag{8}$$

$$MSE(y_{\text{sh}}^{st}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_{\text{sial}}^2 S_{xh}^2 - 2R_{\text{sial}} S_{yhx}) \tag{9}$$

$$MSE(y_{\text{upsi}_1}^{st}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_{\text{upsi}_1}^2 \beta_{2h}^2(x) S_{xh}^2 - 2R_{\text{upsi}_1} \beta_{2h}(x) S_{yhx}) \tag{10}$$

$$MSE(y_{\text{upsi}_2}^{st}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_{\text{upsi}_2}^2 C_{xh}^2 S_{xh}^2 - 2R_{\text{upsi}_2} C_{xh} S_{yhx}) \tag{11}$$

Where, $R_{\text{sidw}} = \frac{\bar{Y}}{\sum_{h=1}^L W_h (\bar{X}_h + C_{xh})}$, $R_{\text{sh}} = \frac{\bar{Y}}{\sum_{h=1}^L W_h (\bar{X}_h + \beta_{2h}(x))}$, $R_{\text{upsi}_1} = \frac{\bar{Y}}{\sum_{h=1}^L W_h (\bar{X}_h \beta_{2h}(x) + C_{xh})}$

$$R_{\text{upsi}_2} = \frac{\bar{Y}}{\sum_{h=1}^L W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))}$$

Tailor et al. (2011) suggested a generalized ratio-cum-product estimator for population mean in stratified random sampling.

$$\bar{y}_5 = \bar{y}_{st} \left[\alpha \left\{ \frac{\sum_{h=1}^L W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))}{\sum_{h=1}^L W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))} \right\} + (1-\alpha) \left\{ \frac{\sum_{h=1}^L W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))}{\sum_{h=1}^L W_h (\bar{X}_h C_{xh} + \beta_{2h}(x))} \right\} \right] \tag{12}$$

$$Bias(y_{\text{tailor}}) = \frac{1}{X} \sum_{h=1}^k W_h^2 \psi_h C_{xh} [\alpha R_{\text{upsi}_2} C_{xh} S_{xh}^2 + (1-2\alpha) S_{yhx}] \tag{13}$$

$$MSE(y_{\text{tailor}}^{opt}) = \sum_{h=1}^k W_h^2 \psi_h S_{yh}^2 - \frac{\left(\sum_{h=1}^k W_h^2 \gamma_h C_{xh} S_{yhx} \right)^2}{\sum_{h=1}^k W_h^2 \gamma_h C_{xh}^2 S_{xh}^2} \tag{14}$$

Hansen et al. (1946) presented combined ratio and product estimators for estimating the population mean \bar{Y} in stratified random sampling are expressed as

$$\bar{y}_{RC} = \bar{y}_{st} \left(\frac{\bar{X}}{x_{st}} \right) \tag{15}$$

$$\bar{y}_{PC} = \bar{y}_{st} \left(\frac{\bar{z}_{st}}{\bar{Z}} \right) \quad (16)$$

Bias and mean squared error of \bar{y}_{RC} and \bar{y}_{PC} are

$$B(\bar{y}_{PC}) = \frac{1}{X} \sum_{h=1}^L W_h^2 \psi_h (R_1 S_{xh}^2 - S_{yxh}) \quad (17)$$

$$B(\bar{y}_{PC}) = \frac{1}{Z} \sum_{h=1}^L W_h^2 \psi_h S_{yzh} \quad (18)$$

$$MSE(\bar{y}_{RC}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_1^2 S_{xh}^2 - 2R_1 S_{yxh}) \quad (19)$$

$$MSE(\bar{y}_{PC}) = \sum_{h=1}^L W_h^2 \psi_h (S_{yh}^2 + R_2^2 S_{zh}^2 + 2R_2 S_{yzh}) \quad (20)$$

$$\text{Where } R_1 = \frac{\bar{Y}}{\bar{X}} \text{ and } R_2 = \frac{\bar{Y}}{\bar{Z}}$$

Singh et al. (2008) defined t_{Re} and t_{Pe} in stratified random sampling as;

$$t_{Re}^{st} = \bar{y}_{st} \exp \left[\frac{\sum_{h=1}^L W_h (\bar{X}_h - \bar{x}_h)}{\sum_{h=1}^L W_h (\bar{X}_h + \bar{x}_h)} \right] \quad (21)$$

$$t_{Pe}^{st} = \bar{y}_{st} \exp \left[\frac{\sum_{h=1}^L W_h (\bar{z}_h - \bar{Z}_h)}{\sum_{h=1}^L W_h (\bar{z}_h + \bar{Z}_h)} \right] \quad (22)$$

To the first degree of approximation biases and mean squared errors of t_{Re}^{st} and t_{Pe}^{st} are;

$$B(t_{Re}^{st}) = \frac{1}{X} \sum_{h=1}^L W_h^2 \psi_h \left(\frac{3}{8} R_1 S_{xh}^2 - \frac{1}{2} S_{yxh} \right) \quad (23)$$

$$B(t_{Pe}^{st}) = \frac{1}{Z} \sum_{h=1}^L W_h^2 \psi_h \left(-\frac{1}{8} R_2 S_{zh}^2 + \frac{1}{2} S_{yzh} \right) \quad (24)$$

$$MSE(t_{Re}^{st}) = \sum_{h=1}^L W_h^2 \psi_h \left(S_{yh}^2 + \frac{1}{4} R_1 S_{xh}^2 - R_1 S_{yxh} \right) \quad (25)$$

$$MSE(t_{Pe}^{st}) = \sum_{h=1}^L W_h^2 \psi_h \left(S_{yh}^2 + \frac{1}{4} R_2 S_{zh}^2 - R_2 S_{yzh} \right) \quad (26)$$

Taylor et al. (2012) introduced a ratio-cum-product estimator for the population mean Y in stratified random sampling.

$$\hat{Y}_{RP}^{ST} = \bar{y}_{st} \left(\frac{\bar{X}}{\bar{x}_{st}} \right) \left(\frac{\bar{z}_{st}}{\bar{Z}} \right) = \sum_h^L W_h \bar{Y}_h \left(\frac{\sum_h^L W_h \bar{X}_h}{\sum_h^L W_h \bar{x}_h} \right) \left(\frac{\sum_h^L W_h \bar{z}_h}{\sum_h^L W_h \bar{Z}_h} \right) \quad (27)$$

$$MSE(\bar{y}_{RP}^{st}) = \sum_{h=1}^K W_h^2 \psi_h \left(S_{yh}^2 + R_1^2 S_{xh}^2 + R_2^2 S_{zh}^2 - 2R_1 S_{yxh} + 2R_2 S_{yzh} - 2R_1 R_2 S_{xzh} \right) \quad (28)$$

Taylor et al. (2015) proposed a ratio-cum-product estimator for the population mean Y in stratified random sampling, utilizing kurtosis information from auxiliary variables x and z as;

$$\hat{Y}_{RP1}^{ST} = \sum_h^L W_h \bar{Y}_h \left(\frac{\sum_h^L W_h (\bar{X}_h + \beta_{2h}(x))}{\sum_h^L W_h (\bar{x}_h + \beta_{2h}(x))} \right) \left(\frac{\sum_h^L W_h (\bar{z}_h + \beta_{2h}(z))}{\sum_h^L W_h (\bar{Z}_h + \beta_{2h}(z))} \right) \quad (29)$$

$$MSE(\bar{y}_{RP1}^{st}) = \sum_{h=1}^K W_h^2 \Psi_h (S_{yh}^2 + R_{12}^2 S_{xh}^2 + R_{13}^2 S_{zh}^2 - 2R_{12} S_{yxh} + 2R_{13} S_{yzh} - 2R_{12} R_{13} S_{xzh}) \quad (30)$$

2.0 PROPOSED ESTIMATOR

Having studied the Taylor et. al (2015), the following estimator is proposed,

$$\bar{y}_{ARP} = \bar{y}_{st} \left[\exp \left(\frac{\sum_{h=1}^L W_h (\bar{X}_h + \beta_{2h(x)})}{\sum_{h=1}^L W_h (\bar{x}_h + \beta_{2h(x)})} \right) \right]^{\theta_1} \left[\exp \left(\frac{\sum_{h=1}^L W_h (\bar{z}_h + \beta_{2h(z)})}{\sum_{h=1}^L W_h (\bar{Z}_h + \beta_{2h(z)})} \right) \right]^{\theta_2} \quad (31)$$

Where the constants θ_1 and θ_2 can be set to values (0,1,-1) to design various estimators; which may be either real numbers or functions of the known parameters, $\beta_{2h}(x)$ and $\beta_{2h}(z)$ represent coefficients of kurtosis of auxiliary variates x and z in h^{th} stratum respectively.

2.1 Properties (Biases and MSEs) of Proposed Estimator

In this section, the bias and MSE for the estimator is derived and discussed. To derive the properties of the estimator, the following error terms are defined:

$$\bar{y}_h = \bar{Y} (1 + \varepsilon_{yh}), \quad \bar{x}_h = \bar{X} (1 + \varepsilon_{xh}) \quad \text{and} \quad \bar{z}_h = \bar{Z} (1 + \varepsilon_{zh})$$

$$E(\varepsilon_{xh}) = E(\varepsilon_{yh}) = E(\varepsilon_{zh}) = 0,$$

$$E(\varepsilon_{xh}^2) = \frac{1}{\bar{X}^2} \sum W_h^2 \Psi_h S_{xh}^2, \quad E(\varepsilon_{yh}^2) = \frac{1}{\bar{Y}^2} \sum W_h^2 \Psi_h S_{yh}^2 \quad \text{and} \quad E(\varepsilon_{zh}^2) = \frac{1}{\bar{Z}^2} \sum W_h^2 \Psi_h S_{zh}^2$$

$$E(\varepsilon_{yh} \varepsilon_{xh}) = \Psi_h \rho_{yxh} C_{yh} C_{xh} \frac{\Psi_h S_{yxh}}{\bar{Y}_h \bar{X}_h}, \quad E(\varepsilon_{xh} \varepsilon_{zh}) = \Psi_h \rho_{xzh} C_{xh} C_{zh} \frac{\Psi_h S_{xzh}}{\bar{X}_h \bar{Z}_h}, \quad E(\varepsilon_{yh} \varepsilon_{zh}) = \Psi_h \rho_{yzh} C_{yh} C_{zh} \frac{\Psi_h S_{yzh}}{\bar{Y}_h \bar{Z}_h}$$

Express (31) in terms of error terms, we have;

$$\bar{y}_{ARP} = \sum W_h \bar{Y}_h (1 + \varepsilon_{yh}) \left[\exp \left(\frac{\sum W_h (\bar{X}_h + \beta_{2h}(x))}{\sum W_h (\bar{X}_h (1 + \varepsilon_{xh}) + \beta_{2h}(x))} \right) \right]^{\theta_1} \left[\exp \left(\frac{\sum W_h (\bar{Z}_h (1 + \varepsilon_{zh}) + \beta_{2h}(z))}{\sum W_h (\bar{Z}_h + \beta_{2h}(z))} \right) \right]^{\theta_2} \quad (32)$$

$$\bar{y}_{ARP} = \bar{Y} + \bar{Y} \varepsilon_{yh} \left[\exp \left(\frac{\sum W_h (\bar{X}_h + \beta_{2h}(x))}{\sum W_h (\bar{X}_h (1 + \varepsilon_{xh}) + \beta_{2h}(x))} \right) \right]^{\theta_1} \left[\exp \left(\frac{\sum W_h (\bar{Z}_h + \beta_{2h}(z)) + \bar{Z} \varepsilon_{zh}}{\sum W_h (\bar{Z}_h + \beta_{2h}(z))} \right) \right]^{\theta_2} \quad (33)$$

$$\bar{y}_{ARP} = \bar{Y} + \bar{Y} \varepsilon_{yh} \left[\exp \left(\frac{1}{1 + \frac{\bar{X} \varepsilon_{xh}}{\sum W_h (\bar{X}_h + \beta_{2h}(x))}} \right) \right]^{\theta_1} \left[\exp \left(1 + \frac{\bar{Z} \varepsilon_{zh}}{\sum W_h (\bar{Z}_h + \beta_{2h}(z))} \right) \right]^{\theta_2} \quad (34)$$

Let

$$\alpha_1 = \frac{\sum W_h \bar{X}_h}{\sum W_h (\bar{X}_h + \beta_{2h}(x))} \quad \text{and} \quad \alpha_2 = \frac{\sum W_h \bar{Z}_h}{\sum W_h (\bar{Z}_h + \beta_{2h}(z))}$$

(34) becomes;

$$\bar{y}_{ARP} = \bar{Y} + \bar{Y} \varepsilon_{yh} \left[\exp(1 - \alpha_1 \varepsilon_{xh} + \alpha_1^2 \varepsilon_{xh}^2) \right]^{\theta_1} \left[\exp(1 + \alpha_2 \varepsilon_{zh}) \right]^{\theta_2} \tag{35}$$

$$\bar{y}_{ARP} = (\bar{Y} + \bar{Y} \varepsilon_{yh}) \left[1 - \frac{4}{5} \alpha_1 \varepsilon_{xh} + \alpha_1^2 \varepsilon_{xh}^2 \right]^{\theta_1} \left[1 + \frac{4}{5} \alpha_2 \varepsilon_{zh} + \frac{1}{5} \alpha_2^2 \varepsilon_{zh}^2 \right]^{\theta_2} \tag{36}$$

$$\bar{y}_{ARP} = (\bar{Y} + \bar{Y} \varepsilon_{yh}) \left[1 - \frac{4}{5} \alpha_1 \theta_1 \varepsilon_{xh} + \left(\frac{17 + 8\theta_1}{25} \right) \alpha_1^2 \theta_1 \varepsilon_{xh}^2 + \frac{4}{5} \alpha_2 \theta_2 \varepsilon_{zh} - \frac{16}{25} \alpha_1 \alpha_2 \theta_1 \theta_2 \varepsilon_{xh} \varepsilon_{zh} + \left(\frac{8\theta_2 - 3}{25} \right) \alpha_2^2 \theta_2 \varepsilon_{zh}^2 \right] \tag{37}$$

$$\bar{y}_{ARP} = \bar{Y} + \bar{Y} \left[\begin{aligned} &\varepsilon_{xh} - \frac{4}{5} \alpha_1 \theta_1 \varepsilon_{xh} + \frac{4}{5} \alpha_2 \theta_2 \varepsilon_{zh} + \left(\frac{17 + 8\theta_1}{25} \right) \alpha_1^2 \theta_1 \varepsilon_{xh}^2 + \left(\frac{8\theta_2 - 3}{25} \right) \alpha_2^2 \theta_2 \varepsilon_{zh}^2 \\ &- \frac{16}{25} \alpha_1 \alpha_2 \theta_1 \theta_2 \varepsilon_{xh} \varepsilon_{zh} - \frac{4}{5} \alpha_1 \theta_1 \varepsilon_{yh} \varepsilon_{xh} + \frac{4}{5} \alpha_2 \theta_2 \varepsilon_{yh} \varepsilon_{zh} \end{aligned} \right] \tag{38}$$

Subtract \bar{Y} from both sides of (38) and take the expectation to obtain the bias of \bar{y}_{ARP} as:

$$Bias(\bar{y}_{ARP}) = \bar{Y} \left[\begin{aligned} &\left(\frac{17 + 8\theta_1}{25} \right) \alpha_1^2 \theta_1 \frac{1}{\bar{X}} \sum_{h=1}^L W_h^2 \psi_h S_{xh}^2 + \left(\frac{8\theta_2 - 3}{25} \right) \alpha_2^2 \theta_2 \frac{1}{\bar{Z}} \sum_{h=1}^L W_h^2 \psi_h S_{zh}^2 - \frac{16}{25} \alpha_1 \alpha_2 \theta_1 \theta_2 \frac{1}{\bar{X}\bar{Z}} \sum_{h=1}^L W_h^2 \psi_h \rho_{xzh} S_{xh} S_{zh} \\ &- \frac{4}{5} \alpha_1 \theta_1 \frac{1}{\bar{X}\bar{Y}} \sum_{h=1}^L W_h^2 \psi_h \rho_{xyh} S_{xh} S_{yh} + \frac{4}{5} \alpha_2 \theta_2 \frac{1}{\bar{Y}\bar{Z}} \sum_{h=1}^L W_h^2 \psi_h \rho_{yzh} S_{yh} S_{zh} \end{aligned} \right] \tag{39}$$

Subtract \bar{Y} from both sides of (38), Squaring, then take expectation to obtain the MSE of \bar{y}_{ARP} is obtained as:

$$MSE(\bar{y}_{ARP}) = E \left[\bar{Y} \left(\varepsilon_{yh} - \frac{4}{5} \alpha_1 \theta_1 \varepsilon_{xh} + \frac{4}{5} \alpha_2 \theta_2 \varepsilon_{zh} \right) \right]^2 \tag{40}$$

$$\begin{aligned} MSE(\bar{y}_{ARP}) &= \sum_{h=1}^L W_h^2 \psi_h S_{yh}^2 + \frac{16}{25} \frac{\bar{Y}^2}{\bar{X}^2} \alpha_1^2 \theta_1^2 \sum_{h=1}^L W_h^2 \psi_h S_{xh}^2 + \frac{16}{25} \frac{\bar{Y}^2}{\bar{Z}^2} \alpha_2^2 \theta_2^2 \sum_{h=1}^L W_h^2 \psi_h S_{zh}^2 \\ &- \frac{8}{5} \frac{\bar{Y}}{\bar{X}} \alpha_1 \theta_1 \sum_{h=1}^L W_h^2 \psi_h \rho_{xyh} S_{xh} S_{yh} + \frac{8}{5} \frac{\bar{Y}}{\bar{Z}} \alpha_2 \theta_2 \sum_{h=1}^L W_h^2 \psi_h \rho_{yzh} S_{yh} S_{zh} - \frac{32}{25} \frac{\bar{Y}\bar{Y}}{\bar{X}\bar{Z}} \alpha_1 \alpha_2 \theta_1 \theta_2 \sum_{h=1}^L W_h^2 \psi_h \rho_{xzh} S_{xh} S_{zh} \end{aligned} \tag{41}$$

Let $R_1 = \frac{\bar{Y}}{\bar{X}}$ and $R_2 = \frac{\bar{Y}}{\bar{Z}}$

$$A = \sum_{h=1}^L W_h^2 \psi_h S_{yh}^2 = V(\bar{y}_{st})$$

$$B = \sum_{h=1}^L W_h^2 \psi_h S_{xh}^2,$$

$$C = \sum_{h=1}^L W_h^2 \psi_h S_{zh}^2,$$

$$D = \sum_{h=1}^L W_h^2 \psi_h \rho_{xyh} S_{yh} S_{xh},$$

$$E = \sum_{h=1}^L W_h^2 \psi_h \rho_{yzh} S_{yh} S_{zh} \quad \text{and} \quad F = \sum_{h=1}^L W_h^2 \psi_h \rho_{xzh} S_{xh} S_{zh}$$

$$\rho_{xyh} = \frac{S_{xyh}}{S_{xh}S_{yh}}, \rho_{yzh} = \frac{S_{yzh}}{S_{yh}S_{zh}} \text{ and } \rho_{xzh} = \frac{S_{xzh}}{S_{xh}S_{zh}}$$

$$MSE(\bar{y}_{ARP}) = A + \frac{16}{25}R_1^2\alpha_1^2\theta_1^2B + \frac{16}{25}R_2^2\alpha_2^2\theta_2^2C - \frac{8}{5}R_1\alpha_1\theta_1D + \frac{8}{5}R_2\alpha_2\theta_2E - \frac{32}{25}R_1R_2\alpha_1\alpha_2\theta_1\theta_2F \tag{42}$$

2.1.1 Minimum MSEs of \bar{y}_{ARP}

The minimum MSE of at optimum values of \bar{y}_{ARP} and, is given by θ_1 and θ_2

To obtain the minimum MSE, \bar{y}_{ARP} is partially differentiated with respect to θ_1 and θ_2 respectively and equate to zero as:

$$\frac{\partial MSE(\bar{y}_{ARP})}{\partial \theta_1} = \frac{\partial MSE(\bar{y}_{ARP})}{\partial \theta_2} = 0 \tag{43}$$

$$\theta_1 = \frac{\frac{5}{4}D + R_2\alpha_2\theta_2F}{R_1\alpha_1B} \tag{44}$$

$$\theta_2 = \frac{R_1\alpha_1\theta_1F - \frac{5}{4}E}{R_2\alpha_2C} \tag{45}$$

Substitute (45) in ((44), we have

$$\theta_{1opt} = \frac{5(DC - EF)}{4R_1\alpha_1(BC - F^2)} \tag{46}$$

Put (44) into (45), we have

$$\theta_{2opt} = \frac{5((DF - BE))}{4R_2\alpha_2(BC - F^2)} \tag{47}$$

Substitute (46) and (47) in (42), the minimum MSE of \bar{y}_{ARP} is obtained as

$$MSE(\bar{y}_{ARP})_{min} = A + \frac{4}{5}R_1\alpha_1 \frac{(DC - EF)}{(BC - F^2)}B + \frac{4}{5}R_2\alpha_2 \frac{(DF - BE)}{(BC - F^2)}C - 2 \frac{(DC - EF)}{(BC - F^2)}D$$

$$+ 2 \frac{(DF - BE)}{(BC - F^2)}E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2}F \tag{48}$$

3.0 THEORETICAL EFFICIENCY COMPARISON

The following section outlines the conditions in which the newly proposed estimator outperforms both the modified estimators from Tailor et al., (2015) and the traditional estimators:

3.1 Usual Unbiased Estimator

Recall

$$V(\bar{y}_{st}) = \sum_{h=1}^L W_h \psi_h S_{yh}^2 = A$$

$MSE(\bar{y}_{ARP})_{min} < MSE(\bar{y}_{st})$ If and only if:

$$\begin{aligned} & \frac{4}{5} R_1 \alpha_1 \frac{(DC - EF)}{(BC - F^2)} B + \frac{4}{5} R_2 \alpha_2 \frac{(DF - BE)}{(BC - F^2)} C - 2 \frac{(DC - EF)}{(BC - F^2)} D \\ & + 2 \frac{(DF - BE)}{(BC - F^2)} E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} F < 0 \end{aligned} \tag{49}$$

3.2 Combined ratio estimator \hat{Y}_{RC}^{ST}
 $MSE(\bar{y}_{ARP}) - MSE(\bar{y}_{RC}) < 0$ If,

$$\begin{aligned} & A + \frac{4}{5} R_1 \alpha_1 \frac{(DC - EF)}{(BC - F^2)} B + \frac{4}{5} R_2 \alpha_2 \frac{(DF - BE)}{(BC - F^2)} C - 2 \frac{(DC - EF)}{(BC - F^2)} D \\ & + 2 \frac{(DF - BE)}{(BC - F^2)} E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} F - (A - R_1^2 B + 2R_1 D) < 0 \end{aligned} \tag{50}$$

3.2 Combined Product Estimator \hat{Y}_{PC}^{ST}
 $MSE(\bar{y}_{ARP}) - MSE(\bar{y}_{PC}) < 0$ If,

$$\begin{aligned} & \frac{4}{5} R_1 \alpha_1 \frac{(DC - EF)}{(BC - F^2)} B + \left(\frac{4}{5} \alpha_2 \frac{(DF - BE)}{(BC - F^2)} - R_2 \right) R_2 C - 2 \frac{(DC - EF)}{(BC - F^2)} D \\ & + \left(\frac{(DF - BE)}{(BC - F^2)} - R_2 \right) 2E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} F < 0 \end{aligned} \tag{51}$$

3.3 Kadilar and Cingi (2003) Ratio Type Estimator \hat{Y}_{SEP}^{ST} If,

$$\begin{aligned} & \left(\frac{4}{5} R_1 \alpha_1 \frac{(DC - EF)}{(BC - F^2)} - R_{1SE}^2 \right) B + \frac{4}{5} R_2 \alpha_2 \frac{(DF - BE)}{(BC - F^2)} C - \left(\frac{(DC - EF)}{(BC - F^2)} - R_{1SE} \right) 2D \\ & + 2 \frac{(DF - BE)}{(BC - F^2)} E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} F < 0 \end{aligned} \tag{52}$$

3.4 Kadilar and Cingi (2003) Product Type Estimator of \hat{Y}_{SEP}^{ST}
 $MSE(\bar{y}_{ARP}) - MSE(\bar{y}_{SEP}) < 0$ If,

$$\begin{aligned} & \frac{4}{5} R_1 \alpha_1 \frac{(DC - EF)}{(BC - F^2)} B + \left(\frac{4}{5} R_2 \alpha_2 \frac{(DF - BE)}{(BC - F^2)} - R_{2SE}^2 \right) C - 2 \frac{(DC - EF)}{(BC - F^2)} D \\ & + \left(\frac{(DF - BE)}{(BC - F^2)} - R_{2SE} \right) 2E - 2 \frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} F < 0 \end{aligned} \tag{53}$$

3.5 Tailor et al. (2012) Estimator \hat{Y}_{RP}^{ST}
 $MSE(\bar{y}_{ARP}) - MSE(\bar{y}_{RP}) < 0$ If,

$$\left(\frac{4}{5}\alpha_1 \frac{(DC - EF)}{(BC - F^2)} - R_1\right)R_1B + \left(\frac{4}{5}\alpha_2 \frac{(DF - BE)}{(BC - F^2)} - R_2\right)R_2C - \left(\frac{(DC - EF)}{(BC - F^2)} - R_1\right)2D + \left(\frac{(DF - BE)}{(BC - F^2)} - R_2\right)2E - \left(\frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} - R_1R_2\right)2F < 0 \tag{54}$$

3.6 Tailor et al. (2015) Estimator \hat{Y}_{RP1}^{ST}

$$MSE(\bar{y}_{ARP}) - MSE(\bar{y}_{RP1}) < 0$$

$$\left(\frac{4}{5}\alpha_1 \frac{(DC - EF)}{(BC - F^2)} - R_{1SE}R_1\right)B + \left(\frac{4}{5}\alpha_2 \frac{(DF - BE)}{(BC - F^2)} - R_{2SE}R_2\right)C - \left(\frac{(DC - EF)}{(BC - F^2)} - R_{1SE}\right)2D + \left(\frac{(DF - BE)}{(BC - F^2)} - R_{2SE}\right)2E - \left(\frac{(DC - EF)(DF - BE)}{(BC - F^2)^2} - R_{1SE}R_{2SE}\right)2F < 0 \tag{56}$$

Equations (49)-(56) provide conditions where the proposed estimator outperforms the unbiased estimator, combined ratio estimator \hat{Y}_{RC}^{ST} , Tailor et al., (2012) estimator \hat{Y}_{RP}^{ST} , and Tailor et al., (2015) estimator \hat{Y}_{RP1}^{ST} .

4.0 EMPIRICAL STUDY

Population I [Source: Murthy (1967)]

Y: Output, X: Fixed capital, Z: Number of workers.

Population 1 [Source: National Horticulture Board]

N=10, n=5

Population 2 [Source: National Horticulture Board (2010), page 208] on All India area, Production and productivity of Cashew-nut.

$n_1 = 2, n_2 = 2$	$N_1 = 5, N_2 = 5$	$S_{yz1} = -411.16$	$S_{yz2} = -1536.24$
$\bar{X}_1 = 214.40$	$\bar{Y}_1 = 1925.80$	$\bar{Z}_1 = 51.80$	$S_{yx1} = 39360.68$
$\bar{X}_2 = 333.80$	$\bar{Y}_2 = 3115.60$	$\bar{Z}_2 = 60.60$	$S_{yx2} = 22356.50$
$S_{x1} = 74.87$	$S_{y1} = 615.92$	$S_{z1} = 0.75$	$S_{zx1} = -38.08$
$S_{x2} = 66.35$	$S_{y2} = 340.38$	$S_{z2} = 4.84$	$S_{zx2} = -287.92$

Y: Productivity (MT/ Hectare), X: Production in ‘000 Tons, Z: Area in ‘000 Hectare

N=10, N1=6, N2=4, n=4, n1=2, n2=2

Table 1: MSEs and PREs of Proposed and Existing Estimators using Pop. I

Estimators	MSE	PRE
\bar{y}_{st}	37141.2	100
\hat{Y}_{RC}^{ST}	119383.7	31.11077
\hat{Y}_{PC}^{ST}	25382.91	146.3236

\hat{Y}_{SER}^{ST}	118232.4	31.41373
\hat{Y}_{SEP}^{ST}	25347.69	146.527
\hat{Y}_{RP}^{ST}	419159.3	8.86088
\hat{Y}_{RP1}^{ST}	184201.6	20.16334
\bar{y}_{ARP}	8573.731	433.1977

From Table 1, the proposed estimator outperforms the conventional unbiased estimator, the combined ratio and product estimators, Kadilar and Cingi's (2003) estimator, Tailor et al., (2012) estimator, and Tailor et al., (2015) estimator in the terms of lower mean squared error and higher relative efficiency.

Table 2: MSEs and PREs of Proposed and Existing Estimators using Pop. II

Estimators	MSE	PRE
\bar{y}_{ST}	0.114516	100
\hat{Y}_{RC}^{ST}	0.1156092	99.05444
\hat{Y}_{PC}^{ST}	0.216607	52.8681
\hat{Y}_{SER}^{ST}	0.1156821	98.99195
\hat{Y}_{SEP}^{ST}	0.2186503	52.37403
\hat{Y}_{RP}^{ST}	0.2249535	50.90653
\hat{Y}_{RP1}^{ST}	0.2197911	52.10221
\bar{y}_{ARP}	0.1140415	100.4161

Table 2 unequivocally demonstrates that the proposed estimator is superior to all other estimators, with lower MSE and higher relative efficiency (PRE). Therefore, it outperforms the conventional unbiased estimator, combined ratio and product estimators, Kadilar and Cingi (2003) estimator, as well as Tailor et al., (2012) and Tailor et al., (2015) estimators in terms of its estimation accuracy and efficiency.

5.0 CONCLUSION

This paper has proposed a generalized family of exponential-type estimators that combine ratio and product, employing power transformation to estimate the population mean of the study variable. This estimator makes use of information about the population mean and the coefficient of kurtosis for two auxiliary variables, x and z , in the stratified random sampling. The paper extensively derived the properties of this newly proposed estimator and

compared them to other relevant estimators. The findings, as presented in the tables, clearly demonstrate that the proposed estimator outperforms the other estimators under consideration in terms of efficiency. As a result, it can be concluded that when information regarding the coefficient of kurtosis of the auxiliary variables is available for each stratum, the proposed estimator performs exceptionally well and proves to be more efficient than the alternatives.

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