

## A Study on Modification of Kumaraswamy Distribution Using Generalized Family of Distributions

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### ABSTRACT

This research present new univariate continuous probability distribution based on the New Kumaraswamy-G family the new distribution, termed New Kumaraswamy-Kumaraswamy distribution (NKw-Kw). This distribution is bounded within the unit interval (0, 1) which could be employed to effectively handle large datasets that exhibit asymmetric or skewed characteristics, as well as those with non-monotonic hazard rate functions as against the classical Kumaraswamy distribution. Several properties of the NKw-Kw distribution was derived. Additionally, a Monte Carlo simulation in three different scenarios was used to assess the model parameters using maximum likelihood estimation. From the goodness of fit analysis, it shows that this model has the lowest Akaike Information Criterion (AIC) compared to existing Kumaraswamy models, and it was observed that as the sample size increases, AIC values decreases which indicate the ability of Nkw-Kw model to handled complex datasets.

**Keywords:** New Kumaraswamy-G family, New Kumaraswamy Kumaraswamy distribution, Maximum likelihood estimation, Akaike Information Criterion, Simulation.

### 1.0 INTRODUCTION

The twenty-first century has witnessed considerable progress in modern statistics, especially regarding distribution theory. A vital component of this advancement has been the development and expansion of new statistical models that are tested on real-world datasets to assess their validity. These datasets cover a spectrum of phenomena, ranging from simple to intricate, enabling researchers to examine the adaptability and precision of these new models across different scenarios. Additionally, simulated data can also be utilized to evaluate the models' efficacy.

Modern distribution theory has prompted statisticians and practitioners to suggest new generalized (G) families and to explore their specialized models, which can effectively be employed in various fields, including medicine, reliability engineering, agriculture, survival analysis, demography, and actuarial studies.

This research proposed a new flexible univariate continuous probability distribution named New Kumaraswamy Kumaraswamy distribution (NKw-Kw), using the G-family proposed by El-Morshedy *et al.* (2022) this distribution will serve as an alternative to the classical Kumaraswamy distribution(Kw) which cannot handle complex dataset.

The Kumaraswamy-G (Kw-G) distribution by Cordeiro and de-Castro (2011) is a well-known G-family, Other distributions in a more recent contributions are the new

Weibull-X (NW-X) by Ahmad *et al.*, (2018), the Rayleigh-EOG-X Family by Yahaya and Doguwa (2021), the New Weibull-Odd Fréchet-G Family by Usman *et al.*, (2021), the Generalized exponential distribution by Ahmad *et al.*, (2021), the Inverse Lomax-G by Falgore and Doguwa (2020), the New Generalized Odd Fréchet G (NGOF-G) by Abubakar Sadiq *et al.*,(2023), the new generalized Lindley distribution by Misra & Kaur (2024), among others. Showing a growing interest in these G-family distributions, Das & Ghosh (2022) used (GKw) in modelling environmental data, Alshahrani & Alzahrani (2023) applied (GKw) in financial risk modelling, others are Gupta & Kaur (2021), Kaur & Singh (2020). Kumar & Singh (2024), Kundu & Basu (2023), Nadarajah & Kotz (2019) Shahbaz & Hussian (2024), and Su & Xie (2022).

Corless *et al.* (1996) introduced the Lambert W-function, named after Johann Heinrich Lambert, which is useful for solving various exponential and logarithmic equations in the form of  $W\{x\}$ .

However, the Kumaraswamy distribution, while classical, does not adequately fit complex data, leading to the development of an extended version known as Kumaraswamy-G (Kw-G) by Cordeiro and de-Castro (2011), which is a generalized family derived from the original Kumaraswamy distribution.

Tahir *et al.*, (2020) proposed the New Kumaraswamy-G family of distributions (NKw-G) through a new

generator, presenting a potential alternative to the earlier Kumaraswamy-G family created by Cordeiro and de-Castro. They anticipated that this new family would provide superior performance.

El-Morshedy *et al.*, (2022) introduced another New Kumaraswamy-G family in their work titled "A New Flexible Univariate and Bivariate Family of Distributions for Unit Interval," using a different generator than that proposed by Tahir *et al.* (2020). This new generator also facilitates the creation of alternative G-classes such as beta-G, McDonald-G, Topp-Leone-G, Marshall-Olkin-G, and Transmuted-G, applicable within the bounded unit interval.

Following the development of the NKw-G family, El-Morshedy *et al.* examined the properties of a specific model termed New Kumaraswamy Weibull distribution (NKwW). They employed both simulated and real-life datasets to estimate the model parameters using the maximum likelihood method and found that the NKwW distribution outperformed nearly all generalized Weibull models.

## 2.0 MATERIALS AND METHODS

The CDF and PDF of Kumaraswamy distribution is given in equations (1) and (2) respectively.

$$G(X) = 1 - (1 - x^\alpha)^\beta; \quad (1)$$

$$g(x) = \alpha\beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1}; \quad (2)$$

Where  $0 < x < 1$ ,  $\alpha, \beta > 0$

### 2.1 The CDF And PDF of The Generalised Family Of Distribution

The cdf and the pdf of the generalized family distribution are given in equations (3) and (4) respectively.

$$F_{NKw-G} = 1 - \left\{ 1 - \left[ 1 - \bar{G}(x) \exp[G(x)] \right]^a \right\}^b$$

$a, b > 0$

(3)

Where  $a$  and  $b$  are shape parameters,

$$1 - \bar{G}(x) \exp[G(x)] \text{ is the new generator and } \bar{G}(X) = 1 - G(X)$$

$$f_{NKw-G} = abg(x)[G(x)] \left[ 1 - \bar{G}(X) \exp[G(x)] \right]^{a-1} \exp[G(x)] \left\{ 1 - \left[ 1 - \bar{G}(X) \exp[G(x)] \right]^a \right\}^{b-1}$$

(4)

Hence, a random variable  $X$  with the density above is denoted by  $X \sim NKw-G(a, b)$

### 2.2 Proposed New Kumaraswamy-Kumaraswamy Distribution (NKw-NKw)

Consider the distribution in (5).

$$\bar{G}(X) = 1 - \left[ 1 - (1 - x^\alpha)^\beta \right] = (1 - x^\alpha)^\beta$$

(5)

The CDF of the proposed distribution is obtained by incorporating equations (1) and (5) in (3) as in (6)

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 - \left\{ 1 - \left[ 1 - (1 - x^\alpha)^\beta \exp \left[ 1 - (1 - x^\alpha)^\beta \right] \right]^a \right\}^b$$

(6)

Hence (6) is the CDF of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution.

The PDF of  $F_{NKw-Kw}$  denoted by  $f_{NKw-Kw}$  is obtained by incorporating equations (1), (2), (5), in (4) as in given (7)

$$f_{NKw-Kw}(x; a, b, \alpha, \beta) = ab\alpha\beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} (1 - (1 - x^\alpha)^\beta) \left[ 1 - (1 - x^\alpha)^\beta \exp \left( 1 - (1 - x^\alpha)^\beta \right) \right]^{a-1} \exp \left( 1 - (1 - x^\alpha)^\beta \right) \left\{ 1 - \left[ 1 - (1 - x^\alpha)^\beta \exp \left( 1 - (1 - x^\alpha)^\beta \right) \right]^a \right\}^{b-1}$$

(7)

Hence (7) is the PDF of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution.

### 2.3 Properties Of New Kumaraswamy Kumaraswamy Distribution

The statistical properties that will be discuss here include the quantiles function, moments, moment generating function, survival function and hazard function.

#### 2.3.1 The Survival Function

Mathematically it is obtained in (8).

$$S(x, a, b, \alpha, \beta)_{NKw-Kw} = 1 - F(X)$$

$$S(x, a, b, \alpha, \beta)_{NKw-Kw} = \left\{ 1 - \left[ \frac{1 - (1 - x^\alpha)^\beta}{\exp(1 - (1 - x^\alpha)^\beta)} \right]^a \right\}^b \quad (8)$$

### 2.3.2 The Hazard Function

Mathematically it can be obtained as follows

$$h(x, a, b, \alpha, \beta)_{NKw-Kw} = \frac{f(a, b, \alpha, \beta)_{NKw-Kw}}{S(a, b, \alpha, \beta)_{NKw-Kw}} \quad (9)$$

$$h(x, a, b, \alpha, \beta)_{NKw-Kw} = \frac{ab\alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \times (1-(1-x^\alpha)^\beta)^{a-1} \left[ 1 - (1-x^\alpha)^\beta \exp(1-(1-x^\alpha)^\beta) \right]^{-a-1} \exp(1-(1-x^\alpha)^\beta)}{\left\{ 1 - \left[ 1 - (1-x^\alpha)^\beta \exp(1-(1-x^\alpha)^\beta) \right]^a \right\}^b} \quad (10)$$

(10) represents the hazard function.

### 2.3.3 Linear Representation for the Density Function OF (NKw-Kw)

By using generalized binomial expansion the CDF of the distribution become,

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 - \sum_{i=1}^{\infty} (-1)^i \binom{b}{i} \left[ \frac{1 - (1 - x^\alpha)^\beta}{\exp(1 - (1 - x^\alpha)^\beta)} \right]^{ai} \quad (11)$$

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 - \sum_{i=1}^{\infty} \sum_{j=1}^{ai} (-1)^{i+j} \binom{b}{i} \binom{ai}{j} \left[ \frac{1 - (1 - x^\alpha)^\beta}{\exp(1 - (1 - x^\alpha)^\beta)} \right]^j \quad (12)$$

Using the generalized binomial expansion, (12) is obtained as,

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 - \sum_{k=0}^{\infty} \sum_{s=0}^j \sum_{i=1}^{\infty} \sum_{j=1}^{ai} \frac{(-1)^{i+j+s} j^k}{k!} \binom{b}{i} \binom{ai}{j} \binom{j}{s} \left[ 1 - (1 - x^\alpha)^\beta \right]^{k+s} \quad (13)$$

The reduce form of equation (13) gives (14) as,

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 + \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} \left[ 1 - (1 - x^\alpha)^\beta \right]^{k+s} \quad (14)$$

Where

$$w_{k,s} = \sum_{i=1}^{\infty} \sum_{j=1}^{ai} \frac{(-1)^{i+j+1} j^k}{k!} \binom{b}{i} \binom{ai}{j} \binom{j}{s}$$

Therefore by differentiating (14) we obtained our PDF as (15),

$$f_{NKW-KW}(x; a, b, \alpha, \beta) = \alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} (k+s) \left[1 - (1-x^\alpha)^\beta\right]^{k+s-1} \quad (15)$$

(15) is the PDF of the distribution in linear representation form.

### 2.3.4 The Moment

Mathematically, the moment is represented as,

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \quad (16)$$

Therefore, incorporating equations (15) in (16), (17) is obtained as,

$$\mu'_r = \int_0^{\infty} x^r \alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} (k+s) \left[1 - (1-x^\alpha)^\beta\right]^{k+s-1} dx \quad (17)$$

Using (17), (18) is obtained as,

$$\mu'_r = \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} (k+s) \int_0^{\infty} x^r \alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \left[1 - (1-x^\alpha)^\beta\right]^{k+s-1} dx \quad (18)$$

(18) is the  $r^{\text{th}}$  moment of New Kumaraswamy Kumaraswamy (NKw-Kw) random variable.

### 2.3.5 Moment Generating Function

Mathematically is given as,

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx \quad (19)$$

Putting equation (17) in (21), (22) is obtained as,

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} (k+s) \alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \left[1 - (1-x^\alpha)^\beta\right]^{k+s-1} dx \quad (20)$$

$$M_X(t) = \alpha\beta \sum_{k=0}^{\infty} \sum_{s=0}^j w_{k,s} (k+s) \int_0^{\infty} e^{tx} x^{\alpha-1} (1-x^\alpha)^{\beta-1} \left[1 - (1-x^\alpha)^\beta\right]^{k+s-1} dx \quad (21)$$

(21) represents the moment generating function of the distribution.

### 2.3.6 The Order Statistics

the probability density function (PDF) for the  $i$ -th order statistic is expressed as follows:

$$g_{in}(x) = \frac{n!}{(i-1)!(n-i)!} abg(x) [G(x)] \left[1 - \bar{G}(X) \exp[G(x)]\right]^{i-1} \exp[G(x)] \left\{1 - \left[1 - \bar{G}(X) \exp[G(x)]\right]^a\right\}^{n-i} \quad (22)$$

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} \alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} \times \left(1 - (1-x^\alpha)^\beta\right) \times \sum_{k=0}^{n-i} \left[1 - (1-x^\alpha)^\beta \exp\left(1 - (1-x^\alpha)^\beta\right)\right]^{i-1} \exp\left(1 - (1-x^\alpha)^\beta\right) \left\{1 - \left[1 - (1-x^\alpha)^\beta \exp\left(1 - (1-x^\alpha)^\beta\right)\right]^a\right\}^{n-i} \quad (23)$$

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} ab\alpha\beta x^{\alpha-1} (1-x^\alpha)^{\beta-1} x \left(1 - (1-x^\alpha)^\beta\right) \exp\left[1 - (1-x^\alpha)^\beta\right] x \sum_{k=0}^{n-1} (-1)^k \binom{n-i}{k} \left\{1 - \left[1 - (1-x^\alpha)^\beta \exp\left[1 - (1-x^\alpha)^\beta\right]\right]^a\right\}^{k+i-1} \tag{24}$$

(24) represent the order statistics of the distribution.

### 2.3.7 The Quantile Function

The quantiles function in simple term can be obtained by taking the inverse of cumulative distribution function (CDF) as below.

Let  $u = F_{NKw-Kw}(x; a, b, \alpha, \beta)$  which is the CDF of the distribution,

Where

$$F_{NKw-Kw}(x; a, b, \alpha, \beta) = 1 - \left\{1 - \left(1 - (1-x^\alpha)^\beta \exp\left[1 - (1-x^\alpha)^\beta\right]\right)^a\right\}^b$$

Then

$$1 - u = \left\{1 - \left(1 - (1-x^\alpha)^\beta \exp\left[1 - (1-x^\alpha)^\beta\right]\right)^a\right\}^b \tag{25}$$

Simplifying further we have

$$1 - (1-u)\frac{1}{ab} = \left(1 - (1-x^\alpha)^\beta\right) \exp\left[1 - (1-x^\alpha)^\beta\right] \tag{26}$$

Then using further simplification and applying Lambert W-function the quantiles is obtained as in equation (27),

$$Q(x) = \left(1 - \left(1 - W\left(1 - (1-u)\frac{1}{ab}\right)\right)^\beta\right)^{\frac{1}{\alpha}} \tag{27}$$

## 3.0 RESULTS AND DISCUSSION

**Table 1. Maximum likelihood estimates of the parameters and first four moments of New Kumaraswamy Kumaraswamy distribution (NKw-Kw).**

	n=100	n=200	n=300
a	1.732	1.039	1.452
b	2.257	1.786	4.360
$\alpha$	0.951	2.727	2.404
$\beta$	3.303	10.333	4.721
$\mu_1^1$	0.329	0.424	0.480
$\mu_2^1$	0.124	0.189	0.238
$\mu_3^1$	0.052	0.088	0.122
$\mu_4^1$	0.024	0.042	0.064

Table 1 represents the maximum likelihood estimates of the parameters from three (3) different scenarios that is when sample sizes are 100, 200 and 300 respectively using different parametric values. It also future the moments from first to fourth moment which captured essential information about the expected values, variability, and shape of distributions, of the three different scenarios respectively.

**Table2: Measures of the model fit for NKw-Kw**

	n=100	n=200	n=300
LL	66.763	184.029	299.525
AIC	-125.527	-360.058	-591.049
BIC	-115.106	-346.865	-576.234
HQIC	-121.309	-354.719	-585.120
SE	0.013	0.007	0.005

Table 2 presents several analytical measures of model fit, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), along with the log likelihood and standard errors for each of the three scenarios. As indicated by the results, all three fit measures such as AIC, BIC, and HQIC are negative for both scenarios. While it is unusual for these values to be negative, their negative values signify the efficiency and precision of the models.

### 3.1 Comparison With Other Models

The existing Kumaraswamy distributions in the literature we reviewed primarily focused on the Akaike Information Criterion (AIC) as the measure of model fit. The specific Kumaraswamy variants examined include the log-exponentiated Kumaraswamy (log-EK), the classical Kumaraswamy (Kw), the transmuted

Kumaraswamy (TKw), and the modified Kumaraswamy (MKw).

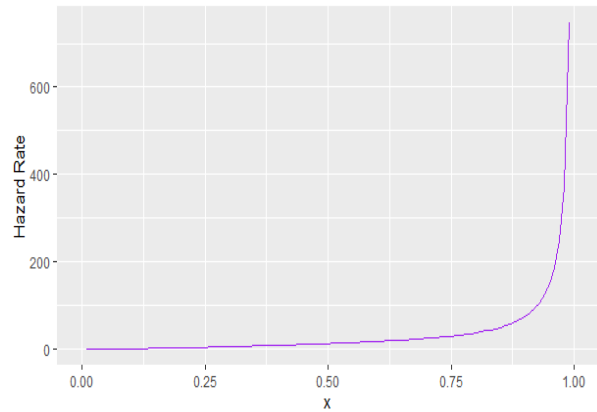
**Table 3. Models Comparison**

Models	AIC
Log-EK	190.55
Kw	-23.078
TKw	-23.995
MKw	-31.775
NKw-Kw	-125.527

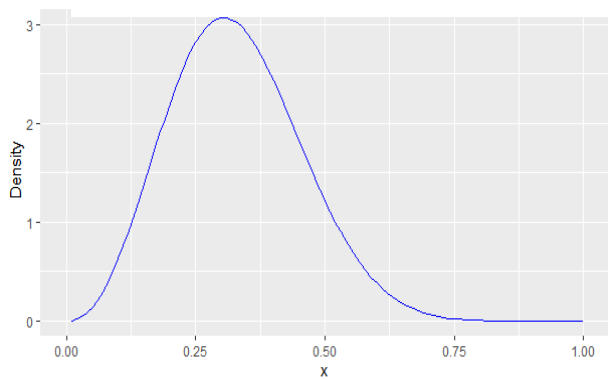
From table 3, we can see that our model, the New Kumaraswamy Kumaraswamy (NKw-Kw) has the least AIC and therefore have a better fit.

### 3.2 Some Graphical Presentations

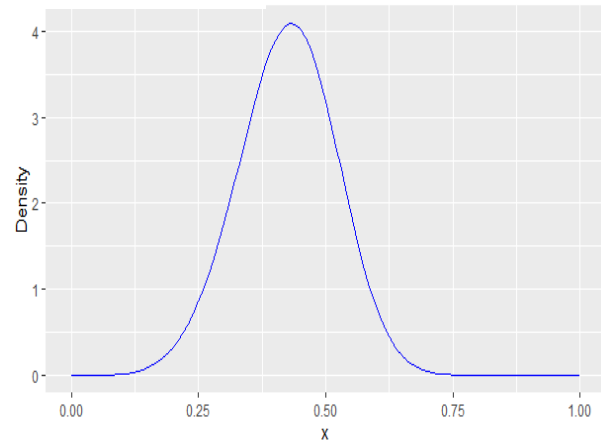
The graphical representation of the PDF, survival and hazard functions of NKw-Kw distribution for three (3) scenarios are presented below.



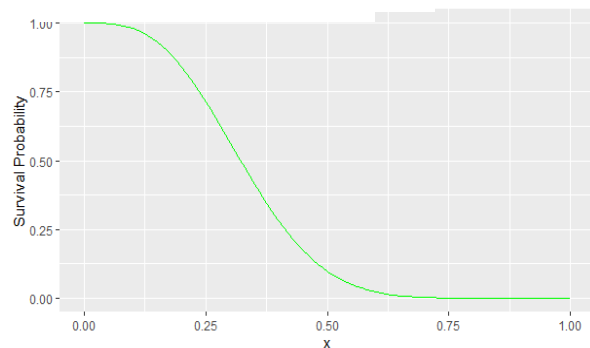
**Figure 3: represent the graph of the hazard function of NKw-Kw distribution at n=100**



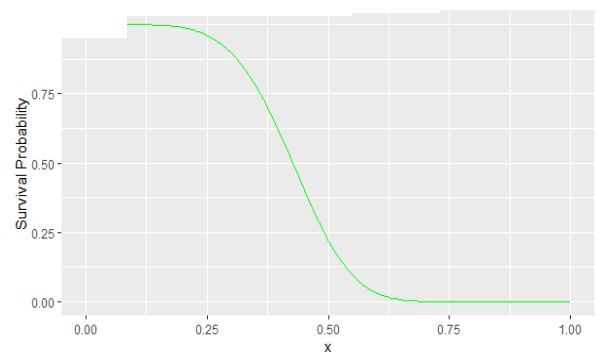
**Figure 1: The graph of PDF of NKw-Kw distribution at n=100**



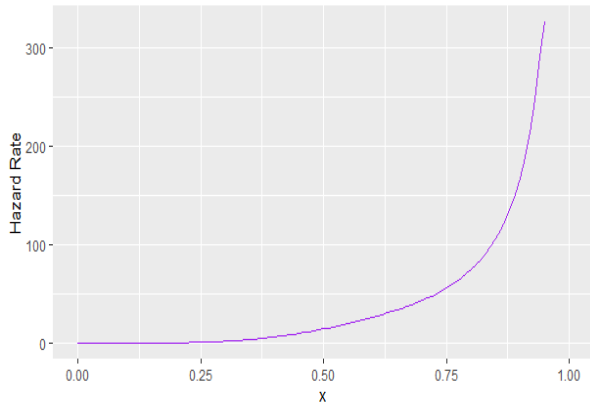
**Figure 4: Represents the graph of the PDF of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=200.**



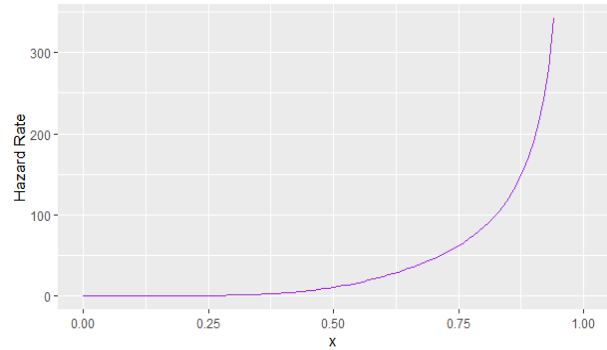
**Figure 2: The graph of the survival function of NKw-Kw distribution at n=100**



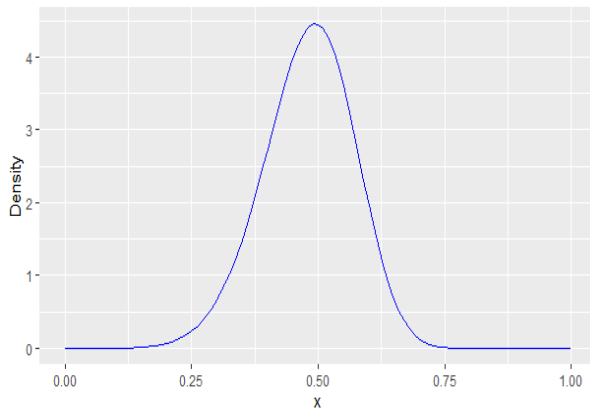
**Figure 4: Represents the graph of the survival function of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=200.**



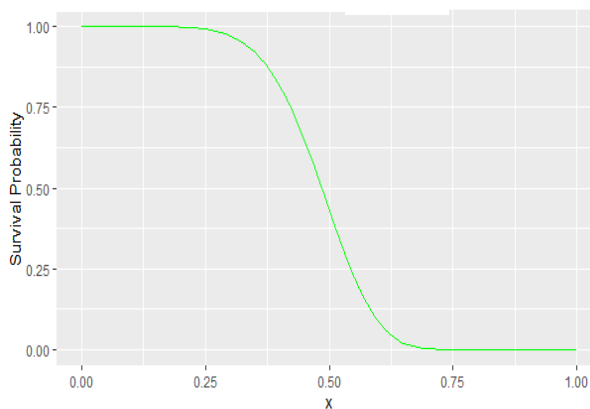
**Figure 4: Represents the graph of the hazard function of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=200.**



**Figure 9: Represents the graph of the hazard function of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=300.**



**Figure 7: Represents the graph of the PDF of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=300.**



**Figure 8: Represents the graph of the survival function of New Kumaraswamy Kumaraswamy (NKw-Kw) distribution at n=300.**

#### 4.0 CONCLUSION

By treating the New Kumaraswamy-G family as a generalized family, we developed a novel continuous univariate probability distribution with four parameters, referred to as the New Kumaraswamy Kumaraswamy distribution (NKw-Kw). This distribution is capable of handling complex datasets, unlike the classical Kumaraswamy distribution. We conducted Monte Carlo studies using three different scenarios to assess the model parameters, moments, and goodness of fit measures such as AIC, BIC, and HQIC. Compared to other existing Kumaraswamy distributions we examined, the NKw-Kw demonstrated superior goodness of fit and increased precision and can graphically takes different shape like left skewed or right skewed among others.

#### AUTHORS CONTRIBUTIONS

The results are part of A.Y.'s dissertation under U.U.'s supervision additionally, authors A.A., A.B., and A.M.N. contributed to the mathematical properties and applications. Furthermore, all authors participated in the preparation and revision of the manuscript.

#### Appendix

Data set 1 n=100

0.39993968 0.21845106 0.35372318  
 0.17352120 0.13550902 0.55810125  
 0.31248634 0.16816779 0.30455685  
 0.33697831 0.12086430 0.33811685  
 0.26087140 0.29734432 0.49931598  
 0.16376939 0.41776551 0.56324142  
 0.38379847 0.12313254 0.16983342  
 0.25537071 0.27398142 0.06028029  
 0.26864653 0.24960581 0.30706137  
 0.28999968 0.39928698 0.46891752  
 0.11444866 0.16226046 0.25613294  
 0.21544102 0.59545599 0.32966674  
 0.23055951 0.43154649 0.38760663



IJSGS FUGUSAU VOL. 10 (4)	WEBSITE: <a href="https://fugus-ijsgs.com.ng">https://fugus-ijsgs.com.ng</a>
0.42435560 0.47158711 0.35173863	0.3850629 0.4781365 0.5217409
0.35203102 0.36829883 0.46568833	0.3443280 0.5654318 0.4374888
0.47410685 0.42370392 0.33374362	0.4254698 0.4013070 0.4745045
0.40904059 0.18683298 0.55771050	0.4316225 0.2479250 0.4331563
0.34199514 0.21397160 0.48534738	0.2961576 0.2813084 0.3988560
0.30132218 0.43637178 0.48149186	0.4526699 0.5383192 0.2658171
0.23258736 0.16662752 0.36622603	0.4838267 0.5880649 0.2547653
0.26531638 0.50581853 0.36274383	0.3654707 0.5404570 0.4152600
0.40545414 0.20714586 0.33979225	0.2483324 0.4053396 0.4543616
0.20915933 0.20813897 0.21591684	0.3876581 0.4782989 0.4818820
0.34282313 0.23212932 0.27791186	0.5100630 0.4640638 0.2023815
0.24899469 0.74865447 0.33051942	0.5352325 0.5669633 0.5406315
0.42976680 0.36426116 0.28360153	0.3750799 0.3958853 0.2955637
0.37466514 0.49306920 0.41887475	0.3802392 0.3600838 0.4228762
0.26427098 0.35063734 0.21849665	0.3841552 0.3289640 0.3428556
0.49936292 0.34455341 0.08377269	0.2119855 0.4448598 0.4806969
0.16779898 0.17158363 0.45275013	0.4530021 0.6596424 0.5232124
0.47937645 0.26956421 0.37780238	0.3199785 0.5061381 0.5034622
0.26827520 0.38674497 0.44597361	0.5761951 0.5012633 0.3617165
0.22094369 0.50685509 0.33346167	0.3178472 0.4292280 0.4589340
0.31813600	0.5010243 0.5556309 0.4583572
	0.4090737 0.5110700 0.4434180
Data set 2 n=200	0.5106846 0.4279455 0.4684598
0.4879732 0.3421023 0.4531383	0.3870480 0.4626019 0.4680226
0.3002244 0.2614294 0.6017148	0.4194877 0.3589995 0.5095976
0.4209960 0.2949768 0.4146753	0.4521081 0.4948042 0.3928324
0.4402238 0.2453539 0.4411075	0.5232205 0.3102024 0.3569044
0.3788639 0.4088820 0.5600650	0.3816492 0.3962363 0.4632939
0.2906172 0.5011433 0.6053420	0.4205288 0.3046429 0.4063714
0.4759288 0.2478931 0.2966162	0.3212940 0.4804756 0.3693925
0.3742201 0.3898020 0.1676991	0.4950168 0.4028812 0.4335895
0.3853723 0.3693167 0.4166771	0.4950121 0.4110868 0.2821465
0.4029371 0.4874885 0.5383108	0.2893716 0.4921294 0.4778108
0.2380647 0.2891112 0.3748656	0.1988276 0.3956763 0.2641383
0.3394024 0.6280636 0.4345280	0.4375550 0.4537252 0.3843346
0.3528316 0.5112405 0.4787814	0.5360297 0.4088178
0.5059805 0.5402296 0.4516167	
0.4518411 0.4642442 0.5359874	
0.5420391 0.5055028 0.4377084	
0.4947134 0.3130133 0.6014390	
0.4441115 0.3380795 0.5500932	
0.4120825 0.5147601 0.5473339	
0.3546087 0.2934551 0.4626720	
0.3825925 0.5646945 0.4600257	
0.4920614 0.3318906 0.4424064	
0.3337238 0.3327956 0.3398301	
0.4447516 0.3542078 0.3930475	
0.3687947 0.7381673 0.4351941	
0.5099404 0.4611796 0.3977194	
0.4690584 0.5556103 0.5019586	
0.3817175 0.4507714 0.3421431	
0.5600984 0.4460878 0.2005701	
0.2946129 0.2983323 0.5266504	
0.5458185 0.3861364 0.4714231	

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