



One Step Hybrid Block Method for The Solution of Optimal Control Problems

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ABSTRACT

This paper considers interpolation and collocation approach for the development of hybrid linear multistep method for the solution of optimal control problems modelled in ordinary differential equations (ODEs). Forward-Backward Sweep method was adopted via Pontryagin's principle. Stability analysis was carried on the developed method and it shows that the developed method is stable, consistent and therefore convergent. Also, numerical examples show that the method compete favourably when compared with the existing method.

Keywords: optimal control, interpolation, collocation, convergent

1.0 INTRODUCTION

Optimal control is a subject where it is desired to determine the inputs to a dynamical system that optimize (i.e., minimize or maximize) a specified performance index while satisfying any constraints on the motion of the system (Rose, 2015). Optimal control models have been used in biological problems, notably in population interactions and diseases, the models range from discrete to systems of ordinary differential equations and partial differential equations, with variable applications such as HIV, vector-borne diseases and pest control (Blayneh and Ngnepieba, 2012). In this research, we consider Optimal Control Problem (OCP) of the form

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] = \int_{t_0}^{t_f} f(x, x(t), u(t)) dt \quad (1)$$

subject to dynamic system

$$\dot{x}' = g(t, x(t), u(t)), \quad (2)$$

$$x(t_0) = x_0, \quad t_f \text{ is free}$$

where $x(t)$ and $u(t)$ are real valued functions, f and g are continuously differentiable functions.

The numerical solution of optimal control problems can be categorized into two different approaches: (1) the direct and (2) the indirect method.

Rose (2015), developed a method for solving optimal control problems, three different methods were considered (Forward Backward Sweep, Shooters method and direct optimization process). The direct methods are based on the transformation of the original optimal control problem into a nonlinear programming (NLP) problem by discretising the state or/and control history and then solving the resulting problem using an NLP solver. The indirect method solves the optimal control problem by deriving the necessary conditions based on Pontryagin's minimum principle.

Ngwane and Jator (2012), work on continuous representation based on the Hybrid Second Derivative Method (HSDM) for the solution of

Initial Value Methods (IVMs) for stiff systems of first order ordinary differential equations by considering the polynomial approximate solution of the form

$$y(x) = \sum_{j=0}^n a_j \varphi^j \quad (3)$$

Interpolating at x_n , collocating at $[x_n, x_{n+\frac{1}{2}}, x_{n+1}]$ to generate the continuous scheme and evaluate at $[x_n, x_{n+1}]$ to generate the discrete scheme which is transform into block. The method is found to be stable and convergent and the numerical results prove the efficiency and competitiveness of the method. Abdelrahim and Umar (2016), developed hybrid block method with three off-step points $[x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}]$ for direct solution of second order ODEs by considering the approximate solution of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x-x_n}{h} \right)^i, i=1,2,\dots,m \quad (4)$$

Interpolating at $[x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}}]$, collocating at $[x_n, x_{n+\frac{1}{5}}, x_{n+\frac{2}{5}}, x_{n+\frac{3}{5}} \text{ and } x_{n+1}]$ and evaluating at $[x_n, x_{n+\frac{1}{5}} \text{ and } x_{n+1}]$ they obtained the hybrid block method of order 5. The method is convergent and absolutely stable with large region of absolute stability.

Rodrigues, Monteiro and Torres (2014), Solve optimal control problems by presenting a numerical approach using the indirect method. The process begins with an initial guess on the control variable. Then, simultaneously, the state equations are solved forward in time and the adjoint equations are solved backward in time. The control is updated by inserting the new values of states and adjoints into its characterization, and the process is repeated until convergence occurs. This is implemented by a well written code for the backward-forward sweep method with fourth order Runge Kutta. This approach can be improved on by considering linear multistep method which will reduce the function evaluation.

Necessary condition for the solution of optimal control problem

(i) State Equation $x = \frac{\partial H}{\partial \lambda} = f(t, x, u)$

(ii) Constrain Equation $\lambda = -\left(\frac{\partial H}{\partial x}\right) = g(t, x, u)$

(iii) Optimal Control Equation $\left(\frac{\partial H}{\partial u} = 0\right) \Rightarrow u = \psi(x, \lambda)$

2.0 DEVELOPMENT OF THE HYBRID BLOCK METHOD

We consider the polynomial approximate solution of the form

$$y(x) = \sum_{n=0}^k a_n x^n \quad (5)$$

first derivative of (5) gives

$$y'(x) = \sum_{n=1}^k n a_n x^{n-1} \quad (6)$$

where $k = s + r - 1$, r and s are the numbers of interpolation and collocation points respectively. Interpolating (5) at x_{n+k} , $k = 1, 2, \dots, r$ and collocating (5) at x_{n+k} , $k = 1, 2, \dots, s$ gives a system of nonlinear equation of the form

$$U = AX \quad (7)$$

where

$$A = [a_0, a_1, a_3, \dots, a_n]$$

$$U = [y_n, y_{n+1}, \dots, y_{n+r}, f_n, f_{n+1}, \dots, f_{n+s}]^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & L & x_n^k \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & L & x_{n+1}^k \\ M & M & M & M & O & M \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & L & x_{n+r}^k \\ 0 & 1 & 2x_n & 3x_n^2 & L & kx_n^{k-1} \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & L & kx_{n+1}^{k-1} \\ M & M & M & M & O & M \\ 0 & 1 & 2x_{n+s} & 3x_{n+s}^2 & L & kx_{n+s}^{k-1} \end{bmatrix}$$

Imposing the following conditions on (5) and (6) respectively

$$y(x_{n+j}) = y_{n+j}, j = 0, 1, 2, 3, \dots, r$$

$$y'(x_{n+j}) = f_{n+j}, j = 0, 1, 2, 3, \dots, s$$

and solving (7) using Cramer's rule and then substituting into (5) gives the continuous nonlinear method.

2.1 Specification of the Method

To generate the hybrid collocation points, we consider arbitrary collocation points. Evaluating (5) at point x_n and (6) at points

$$\left[x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1} \right] \text{ give}$$

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x_{n+\frac{1}{3}}^2 & 4x_{n+\frac{1}{3}}^3 \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x_{n+\frac{2}{3}}^2 & 4x_{n+\frac{2}{3}}^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{2}} \\ f_{n+1} \end{bmatrix}$$

The continuous scheme is given as

$$y_{n+t} = \alpha_0 y_n(t) + \beta_0 f_n(t) + \beta_{\frac{1}{3}} f_{n+\frac{1}{3}}(t) + \beta_{\frac{2}{3}} f_{n+\frac{2}{3}}(t) + \beta_1 f_{n+1}(t) \quad (8)$$

where

$$\alpha_0 = 1, \quad \beta_0 = \frac{1}{8}t(3t-4)(-4t+3t^2+2),$$

$$\beta_{\frac{1}{3}} = \frac{3}{8}t^2(-20t+9t^2+12),$$

$$\beta_{\frac{2}{3}} = -\frac{3}{8}t^2(-16t+9t^2+6),$$

$$\beta_1 = \frac{1}{8}t^2(3t-2)^2$$

we substituted $t = \frac{1}{3}$ in (8)

to obtain the discrete schemes as:

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{8}hf_n + \frac{19}{72}hf_{n+\frac{1}{3}} - \frac{5}{72}hf_{n+\frac{2}{3}} + \frac{1}{72}hf_{n+1} \quad (9)$$

When $t = \frac{2}{3}$

$$y_{n+\frac{2}{3}} = y_n + \frac{1}{9}hf_n + \frac{4}{9}hf_{n+\frac{1}{3}} + \frac{1}{9}hf_{n+\frac{2}{3}} \quad (10)$$

When $t = 1$

$$y_{n+1} = y_n + \frac{1}{8}hf_n + \frac{3}{8}hf_{n+\frac{1}{3}} + \frac{3}{8}hf_{n+\frac{2}{3}} + \frac{1}{8}hf_{n+1} \quad (11)$$

Writing (9), (10) and (11) in block, to get

$$A^{(1)}Y_{m+1} = A^{(0)}Y_m + hB^{(0)}F_m + hB^{(1)}F_{m+1} \quad (12)$$

where

$$Y_{m+1} = \begin{bmatrix} y_{n+\frac{1}{3}} & y_{n+\frac{2}{3}} & y_{n+1} \end{bmatrix}^T,$$

$$Y_m = \begin{bmatrix} y_{n-1} & y_{n-2} & y_n \end{bmatrix}^T$$

$$F_m = \begin{bmatrix} f_{n-1} & f_{n-2} & f_n \end{bmatrix}^T,$$

$$F_{m+1} = \begin{bmatrix} f_{n+\frac{1}{3}} & f_{n+\frac{2}{3}} & f_{n+1} \end{bmatrix}^T$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{9} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ 1 & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

2.2 Stability Properties

We consider the basic properties of the developed method which include order, consistency, zero-stability, convergent and the region of absolute stability.

2.2.1 Order of the Method (Adesanya, Alkali and Sunday, 2014)

Let the linear operator $L\{y(x):h\}$

associated with the block (12) be defined as

$$L\{y(x):h\} = A^{(0)}Y_m - Ey_n - hdf'(y_n) - hbF(Y_m) \quad (13)$$

Expand using Taylor series and comparing the coefficients of h we obtain

$$L\{y(x):h\} = c_0y(x) + c_1hy'(x) + c_2h^2y''(x) + \dots + c_p h^p y^{(p)}(x) + L \quad (14)$$

The linear operator L and the associated continuous linear multistep method (8) are said to be of order p if $c_0 = c_1 = c_2 = \dots = c_p = 0$ and $c_{p+1} \neq 0$.

c_{p+1} is called the error constant and the local truncation error is given by

$$t_{n+k} = c_{p+1}h^{(p+1)}y^{(p+1)}(x_n) + O(h^{r+2}) \quad (15)$$

2.2.2 Consistency

A block method is consistent if it has order $p \geq 1$. Clearly, $p \geq 1$ in this case

2.2.3 Zero-Stability (Adesanya, Alkali and Sunday, 2014)

A block in (12) is said to be Zero-stable, if the roots of the first characteristics polynomial $\rho(z)$ is defined by $\rho(z) = \det(z\tau^{(1)} - \tau^{(0)})$ satisfies $|z| \leq 1$ and the roots $|z| \leq 1$ have multiplicity

not greater than the order of the differential equation.

2.2.4 Convergence

A method is said to be convergent if it is consistent and zero-stable

The developed method is of order 5 and the error constant is given as

$$\eta(z) = \left[-\frac{19}{174960}h^5, -\frac{1}{21870}h^5, -\frac{1}{6480}h^5 \right]^T$$

; it is zero stable, consistent and convergent.

The region of absolute stability of the developed method is shown in Figure 1

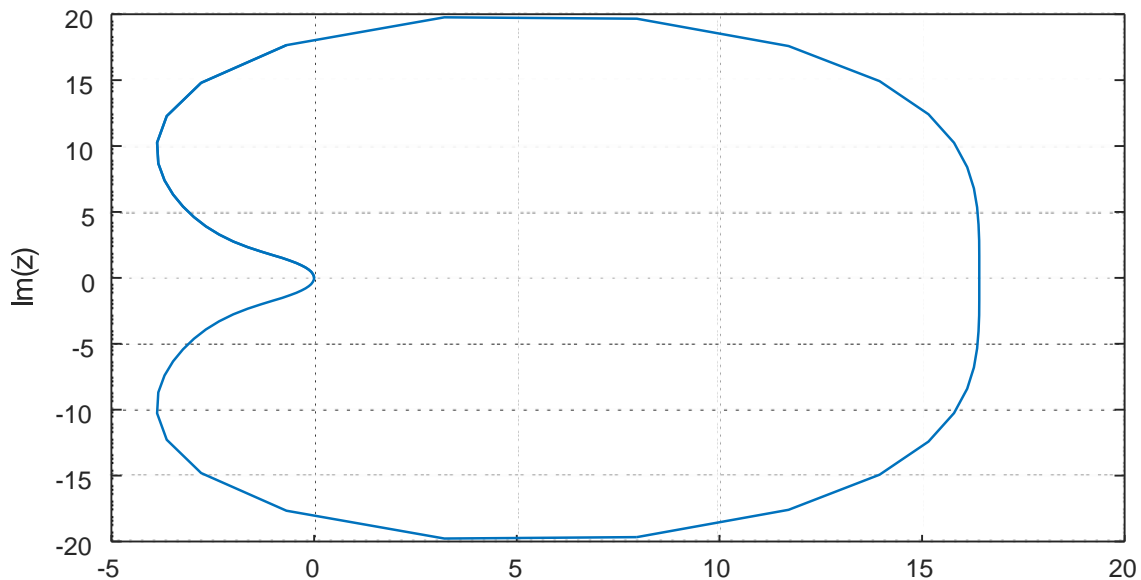


Figure 1: Region of Absolute Stability

3.0 Numerical Examples

Example 1

$$\max_u \int_0^1 x(t) + u(t) dt$$

$$\text{subject to } \begin{cases} x'(t) = 0.5x(t) + u(t), \\ x(0) = 1, \end{cases}$$

with the optimal solution

$$x^*(t) = \frac{2e^{3t} + e^3}{e^{\frac{3t}{2}}(2 + e^3)}, \quad u^*(t) = \frac{2(e^{3t} - e^3)}{e^{\frac{3t}{2}}(2 + e^3)}$$

Source: [7]

$$\min_u \frac{1}{2} \int_0^1 x(t)^2 + u(t)^2 dt$$

$$\text{subject to } \begin{cases} x'(t) = -x(t) + u(t) \\ x(0) = 1 \end{cases}$$

with the optimal solution

$$x(t) = \frac{\sqrt{2} \cosh(\sqrt{2}(t-1)) - \sinh(\sqrt{2}(t-1))}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})},$$

$$\lambda(t) = -\frac{\sinh(\sqrt{2}(t-1))}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})}$$

Source: [1]

Example 2

Example 3

$$\min \int_0^1 x(t) + u(t) dt$$

$$\text{subject to } \begin{cases} x'(t) = 1 - u(t), \\ x(0) = 1, \end{cases}$$

Source: [1]

NOTATIONS

x	Point of Evaluation
HBM	Hybrid Block Method
CRKM	Classical Runge Kutta Method
FBS	Forward Backward Sweep

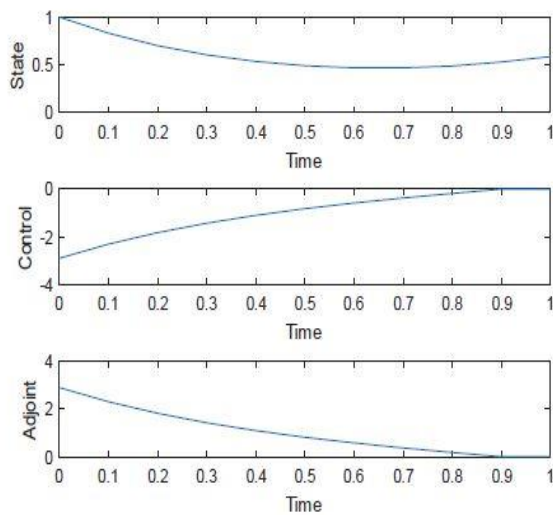


Figure 2: Graphs produced by the FBS for CRKM example 1 at $t = 1, x = 5.8692e-01, u = 0, \lambda = 0.0000$.

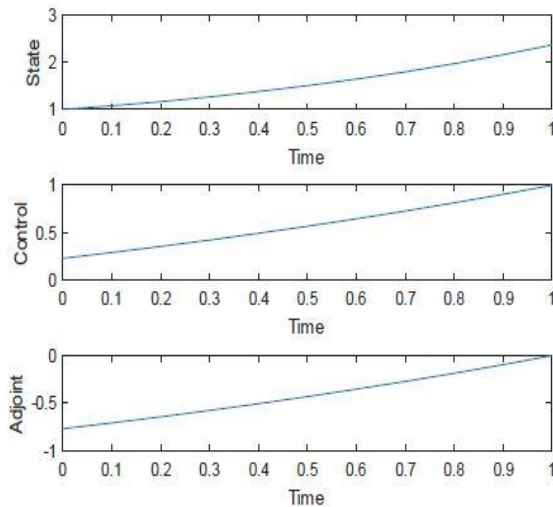


Figure 3: Graphs produced by the FBS for the HBM example 1 at $t = 1, x = 2.3628e+00, u = 0, \lambda = 0.0000$.

$= 9.9902e-01, \lambda = 0.0000$.

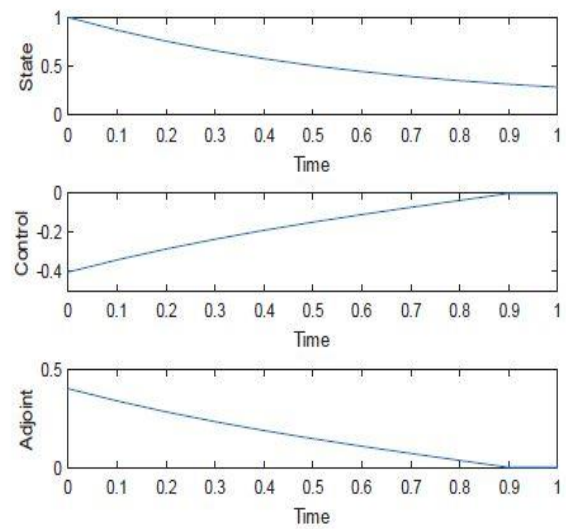


Figure 4: Graphs produced by the FBS for CRKM example 2 at $t = 1, x = 2.8744e-01, u = 0, \lambda = 0.0000$.

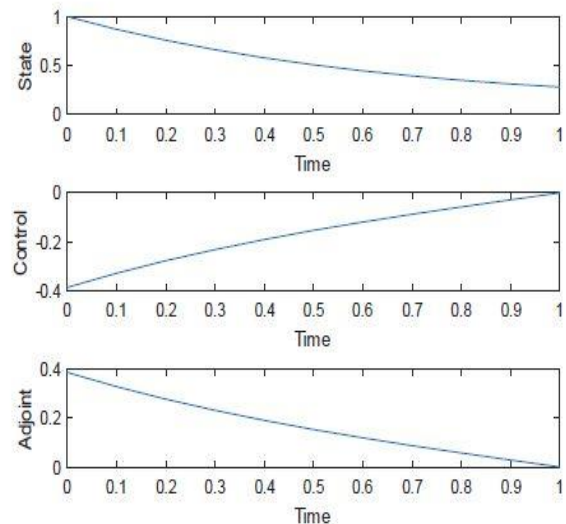


Figure 5: Graphs produced by the FBS for HBM example 2 at $t = 1, x = 2.8177e-01, u = 0, \lambda = 0.0000$.

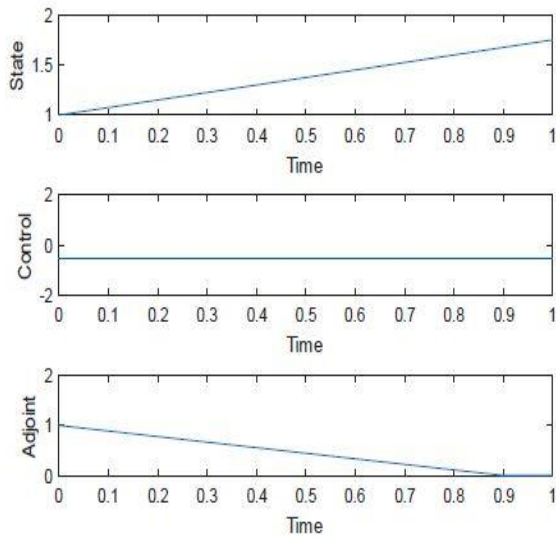


Figure 6: Graphs produced by the FBS for CRKM example 3 at $t = 1$, $x = 1.7510e+00$ $u = 4.9951e-01$ $\lambda = 0.0000$.

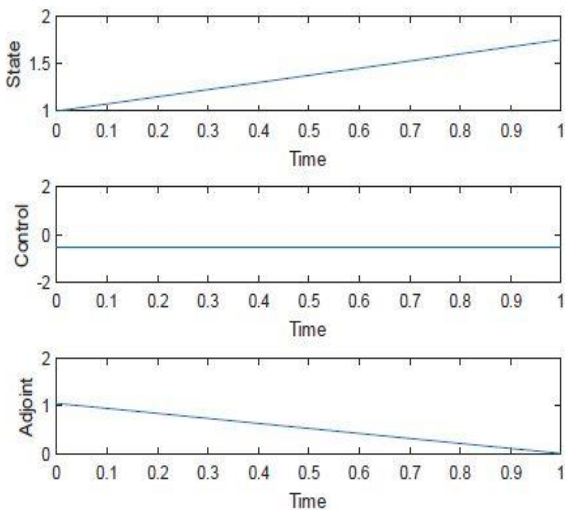


Figure 7: Graphs produced by the FBS for HBM example 3 at $t = 1$, $x = 1.7510e+00$ $u = 4.9951e-01$, $\lambda = 0.0000$

4 CONCLUSIONS

We considered one step method and incorporate two off-grid points for the development of the hybrid block method. The values of x , u , t and λ are taken from Rose (2015) and Lenhart and Workman (2007) for easy comparison. The consistency, convergence, order, error constant and region of absolute stability of the block methods have been determined. The hybrid block method is of order $p = 5$. Figure 1 shows the region of absolute stability. Numerical examples are used

to test the performance of the developed method.

Figure 2 is the results obtained for forward backward sweep method for example 1 of the Classical Runge Kutta method in Lenhart and Workman (2007). Figure 3 depicts the results obtained forward backward sweep method for the developed method for example 1.

Figure 4 illustrates the result of forward backward sweep method for example 2 of the Classical Runge Kutta method in Rose (2015) while Figure 5 shows the results obtained for forward backward sweep method of the developed method for example 2.

Figure 6 represents the results obtained for forward backward sweep method for example 3 of the Classical Runge Kutta method in Rose (2015) while Figure 7 connotes the result of forward backward sweep method of developed method for example 3.

We can therefore, conclude that the results obtained in these figures shows that the new hybrid block method compete effectively compared with [1] and [7].

REFERENCES

- Abdelrahim, R. and Omar, Z. (2016). "Direct Solution of Second-Order Ordinary Differential Equation Using a Single-Step Hybrid Block Method of Order Five". *Mathematical and Computational Applications*, 21(2), p. 1-7.
- Adesanya, A. O. Alkali, A. M. and Sunday, J. (2014). "Order five Hybrid Block Method for the Direct Solution of Second Order Ordinary Differential Equations". *International Journal of mathematical Science and Engineering Applications*. 8(3); p. 285-295.
- Blayneh, K. W. and Ngnepieba, P. (2012). "Optimal control problem for a host-parasitoid model". *International Journal of Pure and Applied Mathematics*, 77(4), p. 565-578.
- Lenhart, S. and Workman, J. T. (2007). "Optimal Control Applied to Biological Models" Chapman & Hall/CRC, London New York.
- Ngwane, F. F. and Jator, S. N. (2012). "Block hybrid second derivative method for stiff

systems”. *International Journal of Pure and Applied Mathematics*, 80(4), p. 543-559.

Rodrigues, H. S., Monteiro, M. T. T. and Torres, D. F. M. (2014). “Systems Theory” *Perspectives, Applications and Developments*. Nova Science Publishers.

Rose, G. R. (2015). “Numerical Methods for Solving Optimal Control Problems”. Tennessee Research and Creative Exchange, University of Tennessee, Knoxville.