

Ascent on Γ_1 Non-Deranged Permutation Group $G_p^{\Gamma_1}$

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ABSTRACT

Some further theoretic properties of the scheme called Γ_1 non-deranged permutation Group, especially in relation to Ascent were identified and studied in this paper. This was done by first computations on this scheme using prime numbers $p \geq 5$. A recursion formula for generating the Ascent number, union of Ascent set, Ascent bottom sum and Ascent top sum was developed and used, these numbers were used to identify theoretic consequences.

Keywords: Ascent Numbers; Ascent set; Γ_1 non-deranged permutation Group; Ascent bottom; Ascent top.

1.0 INTRODUCTION

Permutation statistics were first introduced by Euler in (1913) and then further studied by MacMahon in (1915). In the last decades, much progress has been made, both in the discovery and in the study of new statistics. Also, extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group S_n . Garba and Ibrahim (2010) used the concept to develop a scheme for prime numbers $p \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using $\omega_i = ((1)(1+i)_{mp} (1+2i)_{mp} \dots (1+(p-1)i)_{mp})$ to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non-derangement). The construction of the generated set of permutations from the work of Garba and Ibrahim (2010) as a permutation group was done by Usman and

Ibrahim (2011). However, they achieved this by embedding an identity element into the generated set of permutation (strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as G_p).

With no doubt, patterns in permutations have been well studied for over a century, these patterns were studied on permutations arbitrary. The symmetric group S_n is the set of all permutations of a set Γ of cardinality n . There are several types of other smaller permutation groups (subgroup of S_n) of set Γ , a notable one among them is the alternating group A_n . Afterwards, Ibrahim *et al.* (2016) studied the representation of Γ_1 -non deranged permutation group $G_p^{\Gamma_1}$ via group character, hence established that the character of every $\omega_i \in G_p^{\Gamma_1}$ is never zero. Also, the non standard Young tableaux of Γ_1 -non deranged permutation group $G_p^{\Gamma_1}$ has been studied by Garba *et al.* (2017), they established that the Young tableaux of this permutation group is non standard. Aremu *et*

al.(2017a) studied pattern popularity in Γ_1 -non deranged permutations they establish algebraically that pattern τ_1 is the most popular and pattern τ_3, τ_4 and τ_5 are equipopular in $G_p^{\Gamma_1}$ they further provided efficient algorithms and some results on popularity of patterns of length-3 in $G_p^{\Gamma_1}$ Recently Aremu *et al.*(2017b) studied Fuzzy on Γ_1 -non deranged permutation group $G_p^{\Gamma_1}$ and discover that it is a one sided fuzzy ideal (only right fuzzy but not left) also the α -level cut of f coincides with $G_p^{\Gamma_1}$ if $\alpha = \frac{1}{p}$. Hence, In this paper a recursion formula for generating the Ascent number, union of Ascent set, Ascent bottom sum and Ascent top sum was developed and these numbers were used to identify theoretic consequences.

2.0 PRELIMINARIES

Definition 2.1

Let Γ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset \mathbb{N}$. A bijection ω on Γ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & \dots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \dots & (1+(p-1)i)_{mp} \end{pmatrix}$$

is called a Γ_1 -non deranged permutation.

We denoted G_p to be the set of all Γ_1 -non deranged permutations.

Definition 2.2

The pair G_p and the natural permutation composition forms a group which is denoted as $G_p^{\Gamma_1}$. This is a special permutation group which fixes the first element of Γ .

Definition 2.3

An ascent of permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix}$$

is any positive

$i < n$ (where i and n are positive integers) where the current value is less than the following one, that is i is an ascent of a permutation p if $p_i < p_{i+1}$. The ascent set of p , denoted as $Asc(p)$, is given by $Asc(p) = \{i \in [n-1] : p(i) < p(i+1)\}$ the ascent number of p , denoted as $asc(p)$, is defined as the number of ascent and is given by $asc(p) = |Asc(p)|$.

Definition 2.4

Ascent top is a letter p_i such that $p_i < p_{i+1}$ and the ascent top sum of ω_i is the sum of ascent tops in ω_i , it denoted as $Atop(\omega_i)$.

Definition 2.5

Ascent bottom is a letter p_{i+1} such that $p_i < p_{i+1}$ and the ascent bottom sum of ω_i is the sum of ascent bottoms in ω_i , it denoted as $Abot(\omega_i)$.

Lemma 2.6

It follows from the properties of integer modulo that for $n \in \mathbb{Z}$ an integer $n \pmod{n} = 0$ also implies $n = 0$.

Proof

From definition of integer modulo, we know that

$$n \pmod{n} = 0 \tag{1}$$

It also follows from property of additive identity of \mathbb{Z} that

$$0 \pmod{n} = 0 \tag{2}$$

Now,

equation (1) implies $\frac{n}{n} = 1 + 0$

Equation (2) implies $\frac{0}{n} = 0 + 0$

That is

$\frac{n}{n}$ Is modulo equivalent to

$\frac{0}{n}$ That is $\frac{n}{n} \equiv \frac{0}{n} \equiv 0$ where \equiv represents a notation modulo equivalence.

It follows \equiv is an equivalence relation as we shall establish in the following claim.

Claim

If $\frac{n}{n} \equiv \frac{0}{n}$ Then \equiv is an equivalence relation.

Reflexive

$$\frac{n}{n} \equiv \frac{n}{n}$$

By definition i.e uses $1+0 \equiv 1+0$

$$\text{If } \frac{n}{n} \equiv \frac{0}{n}$$

Then using modulo $\frac{n}{n} = 1+0$

$$\text{And } \frac{0}{n} = 0+0$$

In any case both $\frac{n}{n}$ and $\frac{0}{n} \equiv 0$

$$\text{i.e } \frac{0}{n} \equiv \frac{n}{n}$$

Transitive

$$\text{If } \frac{n}{n} \equiv \frac{0}{n} \text{ And } \frac{0}{n} \equiv \frac{m}{n}$$

It implies that $\frac{n}{n} \equiv \frac{m}{n}$

$$\frac{n}{n} \equiv \frac{0}{n}$$

It implies that $\frac{0}{n} = 0$

$$\text{And } \frac{0}{n} \equiv \frac{m}{n}$$

It implies that $\frac{m}{n} = 0$

It follows $\frac{n}{n} = 0$

From the claim it follows $n = 0$ whenever n is the integer modulo.

For the remainder in this paper we shall adopt the notation established in lemma 2.6 that $n = 0$ for all n representing the modulus of sub sequences in given permutation

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & \dots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \dots & (1+(p-1)i)_{mp} \end{pmatrix}$$

In particular we shall let $n = p$ for some prime integer $p \geq 5$.

3.0 MAIN RESULTS

In this section, we present some ascent results of subgroup $G_p^{\Gamma_1}$ of S_p (Symmetry group of prime order with $p \geq 5$).

Theorem 3.1

The total number of ascent in all Γ_1 -non derangement of $G_p^{\Gamma_1}$ is

$$asc(G_p^{\Gamma_1}) = \frac{1}{2} p(p-1)$$

Proof

If S_p is Symmetry group of degree p and $G_p^{\Gamma_1} \subset S_p$, then the total of position of ascent in S_p is given by

$$\frac{1}{2} p(p-1)p! \tag{3}$$

If $S_p = G_p^{\Gamma_1}$, then equation (3) becomes

$$\frac{1}{2} p(p-1)(p-1) \tag{4}$$

By Γ_1 -non derangement (fixed at 1) of $G_p^{\Gamma_1}$, there will always be an ascent at position 1 with becomes

$$\frac{1}{2} p(p-1)(p-1) + (p-1)(5)$$

Simplifying equation above, we have the total number of ascents to be

$$asc(G_p^{\Gamma_1}) = \frac{1}{2} p(p-1) \tag{5}$$

Lemma 3.2

Let $G_p^{\Gamma_1}$ be a Γ_1 -non deranged permutations then $asc(\omega_i) = p - i$ for all $i \in \{1, 2, 3 \dots p - 1\}$.

Proof

We observe from the computations that the cardinality of the ascent set $asc(\omega_i)$ for all $i \in \{1, 2, 3 \dots p - 1\}$ is $p - i$ since we know that our p is prime and in the properties of the prime, p is one of the factors of the every prime and also by the non derangement properties of $G_p^{\Gamma_1}$. 1 is contained in the ascent set of any arbitrary $\omega_i \in G_p^{\Gamma_1}$. Hence $asc(\omega_i) = p - i$ W

Lemma 3.3

Suppose that $G_p^{\Gamma_1}$ is Γ_1 -non deranged permutations. Then the $Asc(\omega_i) \cap Asc(\omega_{p-1}) = \{1\}$

Proof

Given $\omega_i = a_1 a_2 \dots a_{p-1} a_p$, then $\omega_{p-1} = a_1 a_p a_{p-1} \dots a_2$. From this we observed, since $a_1 = 1$ and it is the least letter and also at the first position of ω_i and ω_{p-1} that

$$Asc(\omega_i) \setminus \{1\} = Des(\omega_{p-1}) \tag{6}$$

and

$$Des(\omega_i) = Asc(\omega_{p-1}) \setminus \{1\} \tag{7}$$

It is obvious from the definitions of descent and ascent that

$$Asc(\omega_i) \cap Des(\omega_i) = \emptyset \tag{8}$$

Substituting (7) into (8) we have

$$Asc(\omega_i) \cap Asc(\omega_{p-1}) \setminus \{1\} = \emptyset$$

$$Asc(\omega_i) \cap Asc(\omega_{p-1}) = \{1\} \tag{W}$$

Lemma 3.4

Suppose that $G_p^{\Gamma_1}$ is Γ_1 -non deranged permutations. Then the $Asc(\omega_i) \cup Asc(\omega_{p-1}) = Asc(\omega_1)$

Proof

Given $\omega_i = a_1 a_2 \dots a_{p-1} a_p$, then

$\omega_{p-1} = a_1 a_p a_{p-1} \dots a_2$ and $\omega_1 = 12 \dots p$. Since ω_1 is a strictly increasing sequence

$$\begin{aligned} Asc(\omega_1) &= \{j : a_j < a_{j+1}\} \cup \{j : a_j > a_{j+1}\} \\ &= Asc(\omega_i) \cup Des(\omega_i) \\ &= Asc(\omega_i) \cup Asc(\omega_{p-1}) \end{aligned}$$

By this we can see that

$$Asc(\omega_i) \cup Asc(\omega_{p-1}) = Asc(\omega_1) \tag{W}$$

Theorem 3.5

The union of Ascent set in all Γ_1 -non derangement of $G_p^{\Gamma_1}$ is

$$\bigcup_{i=1}^{p-1} Asc(\omega_i) = Asc(\omega_1)$$

Proof

Since for any $G_p^{\Gamma_1}$ $\omega_1 = 12 \dots p$ so it take all possible ascent in ω_i , but lemma 3.6 $Asc(\omega_i) \cup Asc(\omega_{p-1}) = Asc(\omega_1)$, so we want to show that for any $G_p^{\Gamma_1}$ there exist ω_i and ω_{p-i} where $i \neq 1$. since $p \geq 5$ and any $G_p^{\Gamma_1}$ consist of non-deranged permutations $\{\omega_1, \omega_2, \omega_3, \dots, \omega_{p-1}\}$, from this at least ω_1 and ω_{p-1} exist W

Theorem 3.6

The Ascent bottom sum in all Γ_1 -non deranged permutations is

$$Abot(\omega_i) = \sum_{k=0}^{p-i-1} (p - k)$$

Proof

By induction for $i = 1$

$$Abot(\omega_i) = p + (p - 1) + \dots + (i + 1)$$

$$Abot(\omega_{i+1}) = p + (p - 1) + \dots + (i + 2)$$

$$Abot(\omega_{i+2}) = p + (p - 1) + \dots + (i + 3),$$

And for any integer $j \geq 0$ such that $i + j < p$, we have

$$Abot(\omega_{i+j}) = p + (p - 1) + \dots + (i + (j + 1))$$

That is,

$$Abot(\omega_{i+j}) = p + (p - 1) + \dots + ((i + j) + 1)$$

Let $i + j = i$

$$Abot(\omega_i) = p + (p - 1) + \dots + (i + 1)$$

Therefore,

$$Abot(\omega_i) = \sum_{k=0}^{p-i-1} (p - k) \text{ W}$$

Theorem 3.7

The Ascent top sum in all Γ_1 -non deranged permutations is

$$Atop(\omega_i) = \sum_{k=1}^{p-i} k$$

Proof

By induction for $i = 1$

$$Atop(\omega_i) = 1 + 2 + \dots + (p + i)$$

$$Atop(\omega_{i+1}) = 1 + 2 + \dots + (p - (i + 1))$$

$$Atop(\omega_{i+2}) = 1 + 2 + \dots + (p - (i + 2)),$$

And for any integer $j \geq 0$ such that $i + j < p$, we have

$$Atop(\omega_{i+j}) = 1 + 2 + \dots + (p - (i + j))$$

Let $i + j = i$. Then

$$Atop(\omega_i) = 1 + 2 + \dots + (p - i),$$

Therefore,

$$Atop(\omega_i) = \sum_{k=0}^{p-i} k \text{ W}$$

Proposition 3.8

If the descent set of a Γ_1 -non derangement ω_{p-i} is X , then the ascent set of $\omega_i = X \cup \{1\}$

Proof

If $\theta_i, \theta_j \in S_p$ (where $i \neq j$ and $i, j \in \mathbb{N}$) then the descent of θ_j is equal to the ascent set of θ_j . By the non derangement property of $G_p^{\Gamma_1}$. 1 is contained in the ascent set of arbitrary $\omega_i \in G_p^{\Gamma_1}$ and so there exist a set $X \cup \{1\}$ that will be ascent set of an arbitrary ω_i W

4.0 CONCLUSION

This paper has provided very useful theoretical properties of this scheme called Γ_1 -non deranged permutations in relation to the Ascent. We have shown that the union of Ascent set of all Γ_1 -non derangement is equal to the identity, we also observed that the difference between $Asc(\omega_i)$ and $Dsc(\omega_{p-1})$ is one.

5.0 RECOMMENDATION

Further researches should be conducted on Γ_1 -non deranged permutations in relation to the other permutation statistics such as Eulerian Statistics, Mahonian Statistics and others in other to find new algebraic and combinatorial results.

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