Ascent on $\Gamma_1$ Non-Deranged Permutation Group $G_p^\Gamma_1$

Ibrahim M., Ibrahim A.A., Garba A.I. and Aremu K.O.
Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346, Sokoto State, Nigeria.
Corresponding Author’s Email: muhammad.ibrahim@udusok.edu.ng

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ABSTRACT

Some further theoretic properties of the scheme called $\Gamma_1$ non-deranged permutation Group, especially in relation to Ascent were identified and studied in this paper. This was done by first computations on this scheme using prime numbers $p \geq 5$. A recursion formula for generating the Ascent number, union of Ascent set, Ascent bottom sum and Ascent top sum was developed and used, these numbers were used to identify theoretic consequences.

Keywords: Ascent Numbers; Ascent set; $\Gamma_1$ non-deranged permutation Group; Ascent bottom; Ascent top.

1.0 INTRODUCTION

Permutation statistics were first introduced by Euler in (1913) and then further studied by MacMahon in (1915). In the last decades, much progress has been made, both in the discovery and in the study of new statistics. Also, extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group $S_n$. Garba and Ibrahim (2010) used the concept to develop a scheme for prime numbers $p \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using

$$\omega_i = ((1)(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp})$$

to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non-derangement). The construction of the generated set of permutations from the work of Garba and Ibrahim (2010) as a permutation group was done by Usman and Ibrahim (2011). However, they achieved this by embedding an identity element into the generated set of permutation (strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as $G_p^\Gamma_1$).

With no doubt, patterns in permutations have been well studied for over a century, these patterns were studied on permutations arbitrary. The symmetric group $S_n$ is the set of all permutations of a set $\Gamma$ of cardinality n. There are several types of other smaller permutation groups (subgroup of $S_n$) of set $\Gamma$, a notable one among them is the alternating group $A_n$. Afterwards, Ibrahim et al. (2016) studied the representation of $\Gamma_1$ non-deranged permutation group $G_p^\Gamma_1$ via group character, hence established that the character of every $\omega_i \in G_p^\Gamma_1$ is never zero. Also, the non standard Young tableaux of $\Gamma_1$ non-deranged permutation group $G_p^\Gamma_1$ has been studied by Garba et al. (2017), they established that the Young tableaux of this permutation group is non standard. Aremu et
al.(2017a) studied pattern popularity in $\Gamma_1$-non deranged permutations they establish algebraically that pattern $\tau_1$ is the most popular and pattern $\tau_3, \tau_4$ and $\tau_5$ are equipopular in $G_p,\Gamma_1$ they further provided efficient algorithms and some results on popularity of patterns of length-3 in $G_p,\Gamma_1$.

Recently Aremu et al. (2017b) studied Fuzzy on $\Gamma_1$-non deranged permutation group $G_p,\Gamma_1$ and discover that it is a one sided fuzzy ideal ( only right fuzzy but not left ) also the $\alpha$−level cut of $f$ coincides with $G_p,\Gamma_1$ if $\alpha = \frac{1}{p}$. Hence, In this paper a recursion formula for generating the Ascent number, union of Ascent set, Ascent bottom sum and Ascent top sum was developed and these numbers were used to identify theoretic consequences.

2.0 PRELIMINARIES

Definition 2.1
Let $\Gamma$ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset \mathbb{N}$. A bijection $\omega$ on $\Gamma$ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 \ldots & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & \ldots & (1+(p-1)i)_{mp} \end{pmatrix}$$

is called a $\Gamma_1$-non deranged permutation.

We denoted $G_p$ to be the set of all $\Gamma_1$-non deranged permutations.

Definition 2.2
The pair $G_p$ and the natural permutation composition forms a group which is denoted as $G_p,\Gamma_1$ . This is a special permutation group which fixes the first element of $\Gamma$.

Definition 2.3
An ascent of permutation $P = \begin{pmatrix} 1 & 2 & 3 \ldots & n \\ p_1 & p_2 & p_3 \ldots & p_n \end{pmatrix}$ is any positive $i < n$(where $i$ and $n$ are positive integers) where the current value is less than the following one, that is $i$ is an ascent of a permutation $p$ if $p_i < p_{i+1}$ . The ascent set of $p$, denoted as $\text{Asc}(p)$, is given by $\text{Asc}(p) = \{ i \in [n-1]: p(i) < p(i+1) \}$ . The ascent number of $p$, denoted as $\text{asc}(p)$, is defined as the number of ascent and is given by $\text{asc}(p) = |\text{Asc}(p)|$.

Definition 2.4
Ascent top is a letter $p_i$ such that $p_i < p_{i+1}$ and the ascent top sum of $\omega_i$ is the sum of ascent tops in $\omega_i$ , it denoted as $\text{Atop}(\omega_i)$.

Definition 2.5
Ascent bottom is a letter $p_{i+1}$ such that $p_i < p_{i+1}$ and the ascent bottom sum of $\omega_i$ is the sum of ascent bottoms in $\omega_i$ , it denoted as $\text{Abot}(\omega_i)$.

Lemma 2.6
It follows from the properties of integer modulo that for $n \in \mathbb{N}$ an integer $n \pmod{n} = 0$ also implies $n = 0$.

Proof
From definition of integer modulo, we know that $n \pmod{n} = 0$ \hspace{1cm} (1)

It also follows from property of additive identity of $\mathbb{N}$ that $0 \pmod{n} = 0$ \hspace{1cm} (2)

Now, equation (1) implies $\frac{n}{n} = 1 + 0$

Equation (2) implies $\frac{0}{n} = 0 + 0$

That $\frac{n}{n}$ is modulo equivalent to
\[
\frac{0}{n} \quad \text{That is } \quad n \equiv 0 \equiv 0
\]
where \( \equiv \) represents a notation modulo equivalence.

It follows \( \equiv \) is an equivalence relation as we shall establish in the following claim.

**Claim**

If \( \frac{n}{n} \equiv \frac{0}{n} \) Then \( \equiv \) is an equivalence relation.

**Reflexive**

\( \frac{n}{n} \equiv \frac{n}{n} \)

By definition i.e uses

\[
1+0 \equiv 1+0
\]

If \( \frac{n}{n} \equiv \frac{0}{n} \)

Then using modulo \( \frac{n}{n} = 1+0 \)

And \( \frac{0}{n} = 0+0 \)

In any case both \( \frac{n}{n} \) and \( \frac{0}{n} \equiv 0 \)

i.e \( \frac{0}{n} \equiv \frac{n}{n} \)

**Transitive**

If \( \frac{n}{n} \equiv \frac{0}{n} \) And \( \frac{0}{n} \equiv \frac{m}{n} \)

It implies that \( \frac{n}{n} \equiv \frac{m}{n} \)

\( \frac{n}{n} \equiv \frac{0}{n} \)

It implies that \( \frac{0}{n} = 0 \)

And \( \frac{0}{n} \equiv \frac{m}{n} \)

It implies that \( \frac{m}{n} = 0 \)

It follows \( \frac{n}{n} = 0 \)

From the claim it follows \( n = 0 \) whenever \( n \) is the integer modulo.

For the remainder in this paper we shall adopt the notation established in lemma 2.6 that \( n = 0 \) for all \( n \) representing the modulus of sub sequences in given permutation

\[
\omega_i = \begin{pmatrix}
1 & 2 & 3 & \ldots & p \\
1 & (1+i)_{mp} & (1+2i)_{mp} & (1+(p-1)i)_{mp}
\end{pmatrix}
\]

In particular we shall let \( n = p \) for some prime integer \( p \geq 5 \).

**3.0 MAIN RESULTS**

In this section, we present some ascent results of subgroup \( G_p^{\Gamma_1} \) of \( S_p \) (Symmetry group of prime order with \( p \geq 5 \)).

**Theorem 3.1**

The total number of ascent in all \( \Gamma_1 \)-non derangement of \( G_p^{\Gamma_1} \) is

\[
\text{asc}(G_p^{\Gamma_1}) = \frac{1}{2} p (p-1)
\]

**Proof**

If \( S_p \) is Symmetry group of degree \( p \) and \( G_p^{\Gamma_1} \subset S_p \), then the total of position of ascent in \( S_p \) is given by

\[
\frac{1}{2} p (p-1)! p!
\]

If \( S_p = G_p^{\Gamma_1} \), then equation (3) becomes

\[
\frac{1}{2} p (p-1)(p-1)
\]

By \( \Gamma_1 \)-non derangement (fixed at 1) of \( G_p^{\Gamma_1} \)
,there will always be an ascent at position 1 with becomes

\[
\frac{1}{2} p (p-1)(p-1) + (p-1)(5)
\]

Simplifying equation above, we have the total number of ascents to be

\[
\text{asc}(G_p^{\Gamma_1}) = \frac{1}{2} p (p-1)
\]
**Lemma 3.2**

Let $G_p^{\Gamma_i}$ be a $\Gamma_1$-non deranged permutations then $\text{asc}(\omega_i) = p - i$ for all $i \in \{1, 2, 3 \ldots p - 1\}$.

**Proof**

We observe from the computations that the cardinality of the ascent set $\text{asc}(\omega_i)$ for all $i \in \{1, 2, 3 \ldots p - 1\}$ is $p - i$ since we know that our $p$ is prime and in the properties of the prime, $p$ is one of the factors of the every prime and also by the non derangement properties of $G_p^{\Gamma_i}$.1 is contained in the ascent set of any arbitrary $\omega_i \in G_p^{\Gamma_i}$. Hence $\text{asc}(\omega_i) = p - i$.

**Lemma 3.3**

Suppose that $G_p^{\Gamma_i}$ is $\Gamma_1$-non deranged permutations. Then the $\text{Asc}(\omega_i) \cap \text{Asc}(\omega_{p-1}) = \{1\}$.

**Proof**

Given $\omega_i = a_1 a_2 \ldots a_{p-1} a_p$, then $\omega_{p-1} = a_1 a_p a_{p-1} \ldots a_2$. From this we observed, since $a_1 = 1$ and it is the least letter and also at the first position of $\omega_i$ and $\omega_{p-1}$ that

$\text{Asc}(\omega_i) \cap \{1\} = \text{Des}(\omega_{p-1})$ (6)

and

$\text{Des}(\omega_i) = \text{Asc}(\omega_{p-1}) \cap \{1\}$ (7)

It is obvious from the definitions of descent and ascent that

$\text{Asc}(\omega_i) \cap \text{Des}(\omega_i) = \emptyset$ (8)

Substituting (7) into (8) we have

$\text{Asc}(\omega_i) \cap \text{Asc}(\omega_{p-1}) \cap \{1\} = \emptyset$

$\text{Asc}(\omega_i) \cap \text{Asc}(\omega_{p-1}) = \{1\}$ \hspace{1cm} \text{W}

**Lemma 3.4**

Suppose that $G_p^{\Gamma_i}$ is $\Gamma_1$-non deranged permutations. Then the $\text{Asc}(\omega_i) \cup \text{Asc}(\omega_{p-1}) = \text{Asc}(\omega_i)$

**Proof**

Given $\omega_i = a_1 a_2 \ldots a_{p-1} a_p$, then $\omega_{p-1} = a_1 a_p a_{p-1} \ldots a_2$. From this we observed, since $\omega_i = 12 \ldots p$. Since $\omega_i$ is a strictly increasing sequence

$\text{Asc}(\omega_i) = \{j : a_j < a_{j+1}\} \cup \{j : a_j > a_{j+1}\}$

$= \text{Asc}(\omega_i) \cup \text{Des}(\omega_i)$

$= \text{Asc}(\omega_i) \cup \text{Asc}(\omega_{p-1})$

By this we can see that $\text{Asc}(\omega_i) \cup \text{Asc}(\omega_{p-1}) = \text{Asc}(\omega_i)$ \text{W}

**Theorem 3.5**

The union of Ascent set in all $\Gamma_1$-non derangement of $G_p^{\Gamma_i}$ is

$p - 1 \bigcup \text{Asc}(\omega_i) = \text{Asc}(\omega_i)$

**Proof**

Since for any $G_p^{\Gamma_i}$, $\omega_i = 12 \ldots p$. So it take all possible ascent in $\omega_i$ but lemma 3.6 $\text{Asc}(\omega_i) \cup \text{Asc}(\omega_{p-1}) = \text{Asc}(\omega_i)$, so we want to show that for any $G_p^{\Gamma_i}$ there exist $\omega_i$ and $\omega_{p-1}$ where $i \neq 1$, since $p \geq 5$ and any $G_p^{\Gamma_i}$ consist of non-deranged permutations $\{\omega_1, \omega_2, \omega_3, \ldots, \omega_{p-1}\}$, from this at least $\omega_i$ and $\omega_{p-1}$ exist \text{W}

**Theorem 3.6**

The Ascent bottom sum in all $\Gamma_1$-non deranged permutations is

$\text{Abot}(\omega_i) = \sum_{k=0}^{p-1} (p - k)$

**Proof**

By induction for $i = 1$

$\text{Abot}(\omega_i) = p + (p - 1) + \ldots + (i + 1)$

$\text{Abot}(\omega_{i+1}) = p + (p - 1) + \ldots + (i + 2)$

$\text{Abot}(\omega_{i+2}) = p + (p - 1) + \ldots + (i + 3)$

And for any integer $j \geq 0$ such that $i + j < p$, we have

$\text{Abot}(\omega_{i+j}) = p + (p - 1) + \ldots + (i + (j + 1))$

That is,

$\text{Abot}(\omega_{i+j}) = p + (p - 1) + \ldots + ((i + j) + 1)$
Let \( i + j = i \)
\[
Abot (\omega_i) = p + (p - 1) + \ldots + (i + 1)
\]
Therefore,
\[
Abot (\omega_i) = \sum_{k=0}^{p-i-1} (p - k) \ W
\]

**Theorem 3.7**
The Ascent top sum in all \( \Gamma_i \)-non deranged permutations is
\[
A_{\top}(\omega_i) = \sum_{k=1}^{p-i} k
\]

**Proof**
By induction for \( i = 1 \)
\[
A_{\top}(\omega_1) = 1 + 2 + \ldots + (p + i)
\]
\[
A_{\top}(\omega_{i+1}) = 1 + 2 + \ldots + (p - (i + 1))
\]
\[
A_{\top}(\omega_{i+2}) = 1 + 2 + \ldots + (p - (i + 2)),
\]
And for any integer \( j \geq 0 \) such that \( i + j < p \), we have
\[
A_{\top}(\omega_{i+j}) = 1 + 2 + \ldots + (p - (i + j))
\]
Let \( i + j = i \). Then
\[
A_{\top}(\omega_i) = 1 + 2 + \ldots + (p - i),
\]
Therefore,
\[
A_{\top}(\omega_i) = \sum_{k=1}^{p-i} k \ W
\]

**Proposition 3.8**
If the descent set of a \( \Gamma_i \)-non derangement \( \omega_{p-i} \) is \( X \), then the ascent set of \( \omega_i = X \cup \{1\} \)

**Proof**
If \( \theta_i, \theta_j \in S_p \) (where \( i \neq j \) and \( i, j \in \mathbb{N} \) ) then the descent of \( \theta_j \) is equal to the ascent set of \( \theta_j \). By the non derangement property of \( G_p^{\Gamma_1} \), if 1 is contained in the ascent set of arbitrary \( \omega_i \in G_p^{\Gamma_1} \), and so there exist a set \( X \cup \{1\} \) that will be ascent set of an arbitrary \( \omega_i \) W

### 4.0 CONCLUSION
This paper has provided very useful theoretical properties of this scheme called \( \Gamma_1 \)-non deranged permutations in relation to the Ascent. We have shown that the union of Ascent set of all \( \Gamma_1 \)-non derangement is equal to the identity, we also observed that the difference between \( A_{\top}(\omega_i) \) and \( D_{\top}(\omega_{p-1}) \) is one.

### 5.0 RECOMMENDATION
Further researches should be conducted on \( \Gamma_1 \)-non deranged permutations in relation to the other permutation statistics such as Eulerian Statistics, Mahonian Statistics and others in order to find new algebraic and combinatorial results.

**References**


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