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**Empirical Estimation of the Parameters of Stochastic Interest Rate
 Models Using Euler – Maruyama Maximum Likelihood Method**

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ABSTRACT

In this paper, we have considered an interest rate model that is driven by Stochastic Differential Equation (SDE) (stochastic interest rate). We employed Euler – Maruyama maximum likelihood method to solve for the unknown parameters analytically. Finally, we estimated the unknown parameters of the SDE from a sample of observations of the stochastic process at known uniformly distributed times using data obtained from Central Bank of Nigeria (CBN) statistical bulletin.

Keywords: Interest rate; maximum likelihood; parameter estimation; likelihood function; financial quantities

1.0 INTRODUCTION

In Financial Mathematics, financial quantities/indicators have been widely modelled using stochastic differential equations (SDEs). Some of such financial quantities/indicators are: interest rates, exchange rate, inflation, asset prices as well as their derivatives. The parameters of the equations are unknown in real applications. Therefore, need to be estimated. For financial models, it is quite a challenge to estimate the model parameters directly from the financial data. Therefore, the estimation of parameters in financial models has become an important research topic and have received more attention in the last decade (De - Rossi, 2010; Tian and Ge, 2012; Platen and Bruti – Liberati, 2010). It is also important to note that estimating the unknown parameters in nonlinear SDEs driven by Wiener processes, when only discrete observation is available is difficult and remains a challenge (Timer, 2000). This is the case because, there exist an unlimited number of solutions for a stochastic differential equation theoretically and it is computationally demanding in obtaining the numerical solutions of SDEs arising from real world application. In Particular, when closed – form solutions for a stochastic differential equation do not exist.

For the past three decades, there has been increasing demand for tools and methods for estimating the parameters of stochastic differential equation in various disciplines i.e. Engineering, Physics, Biology, Finance etc. Here

are some of the parameter estimation methods developed for parameter estimation of SDEs: maximum likelihood estimation (MLE) (Zhenqin, 2014; Fatma and Fathi, 2014; Jungbacker *et al.*, 2011), the methods of moments (Selcuk and Gazanfer, 2014; Otunuga, 2017; Milner *et al.*, 2013) and filtering (Tommaso and Alessandra, 2012; Ionides *et al.*, 2011). Most of the methods developed are for modeling financial quantitative where the financial system displays long time horizons characteristics and often can be sampled at regular but relatively infrequent intervals (Yanli, 2014).

The process of fitting the model to the market data is generally referred to as parameter estimation. Yanli (2014) focuses on parameter calibration for stochastic models. In this work, he proposed an effective algorithm for estimating the parameters in SDE models, which is based on the implementation of the Bayesian inference and the Markov – chain Monte – Carlos (MCMC) method. The author also examined the influence of different samples of the latent variables on the variation of the estimates and suggested that the method used is robust for such variation. He further provided an algorithm for parameter estimation for stiff SDE models. The approach is based on the usage of an implicit numerical scheme in the Monte – Carlo simulation integrated with particle swarm optimization. Experimental results showed that when the SDE model is stiff, estimates produced by the semi – implicit Milstein method have better accuracy

than those obtained by the explicit Milstein method.

On discrete time stochastic models, a Generalized Method of Moments (GMM) (Luck and Wolf, 2016; Otunuga *et al.*, 2017) are developed. This method can be used to compute moment conditions from a discrete time SDE models. The GMM main advantage is that it only requires some moment conditions rather than the full density but since this method does not make use of all information in the sample, it may lead to a loss of efficiency.

Furthermore, maximum likelihood method for direct estimation of parameters of SDEs was proposed based on the Extended Kalman Filter (EKF) and Prediction Error Decomposition (PED) method by (Jiang *et al.*, 2014; Jiang *et al.*, 2017; Wang *et al.*, 2017). The EKF provides state estimates and the recursive residuals, while the PED is used in to calculate estimates of the parameters using Gaussian likelihood function. The computation of the maximum likelihood estimator can be achieved by maximizing the likelihood function associated with a series of observations provided the invariant density and transition density are known explicitly (Barczy and Pap, 2016). However, the transition densities for many realistic and practical useful models, are not available in explicit form. Therefore, complicating the computation of the exact maximum likelihood estimator. But the Euler – Maruyama method can provide tools for approximating the density function.

Modelling financial data that appear as time series using stochastic differential equations, may necessitate the need to estimate the parameters of the equations (may require the estimation of the parameters of the equations). In Econometrics, by maximizing the likelihood function of the sample we can obtain the estimates of optimal parameters (Vaughan, 2014). Therefore, approximate maximum likelihood method provided by Euler Maruyama method provides a very useful tools for this purpose. In other to make the Euler – Maruyama method fairly accurate, the time intervals between all the pairs of consecutive data are split into smaller partition. In this study, we will apply the Euler – Maruyama method to approximate the transition densities of the model equation (interest rate) then estimate the parameters of the interest rate model using data

obtained from CBN statistical bulletin to verify our results.

2.0 THE INTEREST MODEL

The dynamics of the interest rate $r(t)$, is given by the stochastic differential equation described by:

$$dr(t) = (a - br(t))dt + \sqrt{k_1 r(t) + k_2} dW_r(t) \quad (1)$$

$$r(0) = r_0$$

where a, b, k_1 and k_2 are constants.

3.0 DISCRETE TIME MAXIMUM LIKELIHOOD ESTIMATION

An obvious approach to estimating the unknown parameter θ is to replace the continuous time SDE defined in equation (1) by its discrete time approximation. A formal statement of the parameter estimation problem to be addressed is as follows:

$$dX = \mu(X; \theta) + \sigma(X; \theta)dW \quad (2)$$

The task here is to estimate the parameter θ of the SDE from a sample of $(N + 1)$ observations X_0, \dots, X_N of the stochastic process at known uniformly distributed times t_0, \dots, t_N . From equation (2), dW is the differential of the Wiener process, $\mu(X; \theta)$ is the instantaneous drift and $\sigma^2(X; \theta)$ is the instantaneous diffusion.

Among many possible discretization schemes, we chose the Euler – Maruyama discretization scheme. The transition PDF can be approximated by the Gaussian distribution, that is

$$X_{t_{i+1}} | X_{t_i} \sim N(X_{t_i} + \mu(X_{t_i}; \theta)\Delta, \sigma^2(X_{t_i}; \theta)\Delta)$$

Hence, the likelihood function is defined as the product of the probability density functions of each observation of the random variable. Therefore,

$$\begin{aligned} L(\theta|X) &= \prod_{i=1}^n p(X_{t_i} | X_{t_{i-1}}; \theta) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2(X_{t_{i-1}}; \theta)\Delta}} \exp[-Z] \\ &= \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2(X_{t_{i-1}}; \theta)\Delta}} \exp[-Z] \right) \quad (3) \\ Z &= \frac{(X_{t_i} - (X_{t_{i-1}} + \mu(X_{t_{i-1}}; \theta)\Delta))^2}{2\sigma^2(X_{t_{i-1}}; \theta)\Delta} \end{aligned}$$

where

$p(X_{t_i}|X_{t_{i-1}}; \theta) = p(X_{t_i}, t_i|X_{t_{i-1}}, t_{i-1}; \theta)$ is the value of the transitional Probability Density Function (PDF) at (X_{t_i}, t_i) for a process starting at $(X_{t_{i-1}}, t_{i-1})$ and evolving to (X_{t_i}, t_i) in accordance with equation (2)

For easy computation, we solve for the natural logarithm function denoted by $l(\theta|X)$ and optimizing $l(\theta|X)$ is equivalent to optimizing $L(\theta|X)$. Therefore, the log – likelihood function can be obtained by the taking the natural logarithm of both sides of (3). i.e.,

$$\begin{aligned}
 l(\theta|X) &= \log L(\theta|X) \\
 &= -\frac{n}{2} \log(2\pi) \\
 &\quad -\frac{1}{2} \sum_{i=1}^n \log(\sigma^2(X_{t_{i-1}}; \theta)\Delta t) \\
 &\quad -\frac{1}{2} \sum_{i=1}^n \frac{(X_{t_i} - (X_{t_{i-1}} + \mu(X_{t_{i-1}}; \theta)\Delta t))^2}{\sigma^2(X_{t_{i-1}}; \theta)\Delta t} \quad (4)
 \end{aligned}$$

The unknown parameter θ which in this sense a vector can now be solved analytically (if possible) or numerically by setting equation (4) equal to zero. $\Delta t = t_i - t_{i-1}$ is the time step for the process.

4.0 RESULTS AND DISCUSSION

4.1 Estimation of the Interest Rate Model Parameters by Euler – Maruyama Maximum Likelihood Method

Here we consider the SDE given in equation (1) below.

$$\begin{aligned}
 dr(t) &= (a - br(t))dt \\
 &\quad + \sqrt{k_1 r(t) + k_2} dW_r(t) \quad r(t_0) = r_0 \quad (5)
 \end{aligned}$$

Let $k_2 = 0$, then from equation (5) we have

$$\begin{aligned}
 dr(t) &= (a - br(t))dt \\
 &\quad + \sqrt{k_1 r(t)} dW_r(t), \quad r(t_0) = r_0 \quad (6)
 \end{aligned}$$

Let $\sigma_\eta = \sqrt{k_1}$

Then, from equation (6) the Euler – Maruyama discretization of the interest rate model is given by:

$$\begin{aligned}
 r_{t_i} &= r_{t_{i-1}} + (a - br_{t_{i-1}})\Delta t \\
 &\quad + \sigma_\eta \sqrt{r_{t_{i-1}}} \Delta W_i \quad r(t_0) = r_0 \quad (7)
 \end{aligned}$$

Here, $\mu(r_{t_{i-1}}; \theta) = (a - br_{t_{i-1}})$,

$\sigma^2(r_{t_{i-1}}; \theta) = \sigma_\eta^2 r_{t_{i-1}}$, $\theta = (a, b, \sigma_\eta)$ and

$\sigma_\eta = \sqrt{k_1}$.

Therefore, from equation (4) we have that

$$\begin{aligned}
 l(\theta|X) &= -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^n \log(\sigma_\eta^2 r_{t_{i-1}} \Delta t) \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \frac{(r_{t_i} - (r_{t_{i-1}} + (a - br_{t_{i-1}})\Delta t))^2}{\sigma_\eta^2 r_{t_{i-1}} \Delta t} \quad (8)
 \end{aligned}$$

Observe that in the case of equation (8), we can find analytical expression for the unknown parameters that maximize the likelihood function. Hence,

$$\frac{\partial l(\theta|X)}{\partial a} = \frac{1}{\sigma_\eta^2} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)$$

$$\frac{\partial l(\theta|X)}{\partial b} = \frac{1}{\sigma_\eta^2} \sum_{i=1}^n (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)$$

$$\frac{\partial l(\theta|X)}{\partial \sigma_\eta} = -\frac{n}{\sigma_\eta} + \frac{1}{\sigma_\eta^3 \Delta t} \times$$

$$\sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)^2$$

Therefore,

$$U(\theta; X) = \begin{bmatrix} \frac{\partial l(\theta; r)}{\partial a} \\ \frac{\partial l(\theta; r)}{\partial b} \\ \frac{\partial l(\theta; r)}{\partial \sigma_\eta} \end{bmatrix} \quad (9)$$

Next, we set the components of equation (9) to zero. i.e.

$$\begin{aligned}
 \frac{1}{\sigma_\eta^2} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t) &= 0 \quad (10)
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{\sigma_\eta^2} \sum_{i=1}^n (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t) &= 0 \quad (11)
 \end{aligned}$$

$$\begin{aligned}
 -\frac{n}{\sigma_\eta} + \frac{1}{\sigma_\eta^3 \Delta t} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)^2 &= 0 \quad (12)
 \end{aligned}$$

From equation (10), we have

$$\sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t) = 0$$

which is same as

$$\sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} - n - a\Delta t \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} + nb\Delta t = 0$$

or

$$a\Delta t \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} = \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} + nb\Delta t - n$$

Therefore, \hat{a} is estimated as

$$\hat{a} = \frac{\sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} + nb\Delta t - n}{\Delta t \sum_{i=1}^n \frac{1}{r_{t_{i-1}}}} \quad (13)$$

Substituting for \hat{a} in equation (11) gives

$$-\frac{1}{\sigma_\eta^2} \sum_{i=1}^n \left(r_{t_i} - r_{t_{i-1}} - \left(\frac{\sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} + nb\Delta t - n}{\Delta t \sum_{i=1}^n \frac{1}{r_{t_{i-1}}}} - br_{t_{i-1}} \right) \Delta t \right) = 0$$

or

$$\sum_{i=1}^n r_{t_i} - \sum_{i=1}^n r_{t_{i-1}} - \left(\frac{n \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} + n^2 b\Delta t - n^2}{\sum_{i=1}^n \frac{1}{r_{t_{i-1}}}} \right)$$

$$+ \Delta t b \sum_{i=1}^n r_{t_{i-1}} = 0$$

simplifying the above equation gives

$$\sum_{i=1}^n r_{t_i} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} - \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} - n \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} + n^2 + \Delta t b \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} - n^2 b\Delta t = 0$$

or

$$b\Delta t \left(n^2 - \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} \right) = \sum_{i=1}^n r_{t_i} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} + n^2 - n \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} - \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}}$$

$$\hat{b} = \frac{Y}{Q} \quad (14)$$

$$Y = \sum_{i=1}^n r_{t_i} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} + n^2 - n \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}}$$

$$Q = \Delta t \left(n^2 - \sum_{i=1}^n r_{t_{i-1}} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} \right)$$

Also, from equation (12), we have

$$-\frac{n}{\sigma_\eta} + \frac{1}{\sigma_\eta^3 \Delta t} \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)^2 = 0$$

so that

$$n\sigma_\eta^2 \Delta t = \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (a - br_{t_{i-1}})\Delta t)^2$$

$$\sigma_\eta^2 = \frac{R}{n\Delta t}$$

$$R = \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (\hat{a} - \hat{b}r_{t_{i-1}})\Delta t)^2$$

so that

$$\hat{\sigma}_\eta = \sqrt{\frac{R}{n\Delta t}} \quad (15)$$

$$\sigma_\eta = \sqrt{k_1}$$

Therefore,

$$k_1 = \frac{R}{n\Delta t} \quad (16)$$

From equation (16), we have that

$$(r_{t_i} - r_{t_{i-1}} - (\hat{a} - \hat{b}r_{t_{i-1}})\Delta t)^2 = r_{t_i}^2 + r_{t_{i-1}}^2 + \hat{b}^2 r_{t_{i-1}}^2 (\Delta t)^2 + 2\hat{b}r_{t_i}r_{t_{i-1}}\Delta t + 2\hat{a}r_{t_{i-1}}\Delta t + \hat{a}^2 (\Delta t)^2 - 2\hat{b}r_{t_{i-1}}^2 \Delta t - 2r_{t_i}r_{t_{i-1}} - 2\hat{a}r_{t_i}\Delta t - 2\hat{a}\hat{b}r_{t_{i-1}}(\Delta t)^2$$

Therefore,

$$\frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (\hat{a} - \hat{b}r_{t_{i-1}})\Delta t)^2 = \frac{r_{t_i}^2}{r_{t_{i-1}}} + r_{t_{i-1}} + \hat{b}^2 r_{t_{i-1}} (\Delta t)^2 + 2\hat{b}r_{t_i}\Delta t + 2\hat{a}\Delta t + \frac{\hat{a}^2 (\Delta t)^2}{r_{t_{i-1}}} - 2\hat{b}r_{t_{i-1}}\Delta t - 2r_{t_i} - \frac{2\hat{a}r_{t_i}\Delta t}{r_{t_{i-1}}} - 2\hat{a}\hat{b}(\Delta t)^2$$

Hence,

$$\sum_{i=1}^n \frac{1}{r_{t_{i-1}}} (r_{t_i} - r_{t_{i-1}} - (\hat{a} - \hat{b}r_{t_{i-1}})\Delta t)^2 = \sum_{i=1}^n \frac{r_{t_i}^2}{r_{t_{i-1}}} + \sum_{i=1}^n r_{t_{i-1}} + \hat{b}^2 (\Delta t)^2 \sum_{i=1}^n r_{t_{i-1}}$$

$$\begin{aligned}
 &+2\hat{b}\Delta t \sum_{i=1}^n r_{t_i} + \sum_{i=1}^n 2\hat{a}\Delta t \\
 &+ \hat{a}^2(\Delta t)^2 \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} - 2\hat{b}\Delta t \sum_{i=1}^n r_{t_{i-1}} \\
 &- 2 \sum_{i=1}^n r_{t_i} - 2\hat{a}\Delta t \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} \\
 &- \sum_{i=1}^n 2\hat{a}\hat{b}(\Delta t)^2
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{\sigma}_\eta = \text{sqr}t \left(\left(\sum_{i=1}^n \frac{r_{t_i}^2}{r_{t_{i-1}}} + \sum_{i=1}^n r_{t_{i-1}} \right. \right. \\
 \left. \left. + \hat{b}^2(\Delta t)^2 \sum_{i=1}^n r_{t_{i-1}} \right. \right. \\
 \left. \left. + 2\hat{b}\Delta t \sum_{i=1}^n r_{t_i} + n2\hat{a}\Delta t \right. \right. \\
 \left. \left. + \hat{a}^2(\Delta t)^2 \sum_{i=1}^n \frac{1}{r_{t_{i-1}}} - 2\hat{b}\Delta t \sum_{i=1}^n r_{t_{i-1}} \right. \right. \\
 \left. \left. - 2 \sum_{i=1}^n r_{t_i} - 2\hat{a}\Delta t \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}} \right. \right. \\
 \left. \left. - \sum_{i=1}^n 2\hat{a}\hat{b}(\Delta t)^2 \right) / n\Delta t \right) \quad (17)
 \end{aligned}$$

Let

$$P_{01} = \sum_{i=1}^n r_{t_i}, P_{02} = \sum_{i=1}^n r_{t_{i-1}}, P_{03} = \sum_{i=1}^n \frac{r_{t_i}^2}{r_{t_{i-1}}}$$

$$P_{04} = \sum_{i=1}^n \frac{1}{r_{t_{i-1}}}, P_{05} = \sum_{i=1}^n \frac{r_{t_i}}{r_{t_{i-1}}}$$

Then

$$\hat{a} = \frac{P_{05} + n\hat{b}\Delta t - n}{\Delta t P_{04}} \quad (18)$$

$$\hat{b} = \frac{P_{01}P_{04} + n^2 - nP_{05} - P_{02}P_{04}}{\Delta t(n^2 - P_{02}P_{04})} \quad (19)$$

$$\begin{aligned}
 \hat{\sigma}_\eta = \text{sqr}t \left(P_{03} + P_{02} + \hat{b}^2(\Delta t)^2 P_{02} \right. \\
 \left. + 2\hat{b}\Delta t P_{01} + 2n\hat{a}\Delta t + \hat{a}^2(\Delta t)^2 P_{04} \right. \\
 \left. - 2\hat{b}\Delta t P_{02} - 2P_{01} - 2\hat{a}\Delta t P_{05} \right. \\
 \left. - 2n\hat{a}\hat{b}(\Delta t)^2 \right) / n\Delta t \quad (20)
 \end{aligned}$$

From equations (18) to (20), we have that \hat{a} , \hat{b} and $\hat{\sigma}_\eta$ are the approximated parameters using the method of maximum likelihood estimation. The Maximum likelihood estimation method is then applied to the annual series of interest rates obtained from Central Bank of Nigeria (CBN) statistical bulletin. The numerical

results or the estimated parameter values can be found as given in Table 1.

Table 1: Estimated Parameters Value

Parameters	a	b	σ_r
Estimated Parameter Values	0.3754	2.5148	0.3535

5.0 CONCLUSION

In financial mathematics, Stochastic Differential Equations (SDEs) have been widely used to model financial quantities/indicators such as in interest rates, exchange rate, inflation, asset prices as well as their derivatives. In real world applications, the parameters of the equations are unknown and need to be estimated. Therefore, the estimation of parameters in financial models is of great importance. We have considered a stochastic interest rate model and discrete maximum likelihood method was employed to obtain unknown parameters analytically, using data obtained from Central Bank of Nigeria (CBN) statistical bulletin.

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