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Bifurcation Analysis of a Mathematical Model for Tuberculosis Transmission

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ABSTRACT

The total population of the model is divided into five mutually exclusive classes: Vaccinated $V(t)$, susceptible $S(t)$, latent $L_T(t)$, infected $I_T(t)$ and Recovery $R_T(t)$. Invariant region of the model were obtained and the result shows that whenever $N_T > \frac{\Lambda}{N}$, the population reduces asymptotically to the carrying capacity and whenever $N_T \leq \frac{\Lambda}{N}$, every solution with initial condition in Ω remains in that region for $t > 0$, so the model is well posed in Ω . We obtained positivity of Solution, it shows that all the solutions of the system are all positive for all $t > 0$; disease free and endemic equilibrium of the model were obtained, more, also the model threshold parameter (Reproduction Number) was examined using next-generation operator method. The model result shows that diseases free equilibrium is local asymptotically stable at $R_0 < 1$ and unstable at $R_0 > 1$. we also tested to know if the system will exhibits a backward bifurcation and the result show that $\lambda_T = 0$ correspond to disease free equilibrium point (PDE) and $\lambda_T > 0$ correspond to a situation when the disease persists (endemic), then we applied transcritical bifurcation and the result show that $a > 0$ and $b > 0$ hence backward bifurcation; we also obtained Hopf Bifurcation for the system and the result show that $\beta = \bar{\beta}$ and $\frac{dT}{d\beta} \neq 0$ which means that the susceptible, latent and Recovery have to be control or else the endemic will occur and finally sensitivity analysis of R_{0T} with respect to the model parameters were carried out.

Keywords: Transcritical, Bifurcation, hopf Bifurcation, equilibrium, endemic.

1.0 INTRODUCTION

Tuberculosis (also known as "TB") is a disease caused by a type of bacteria called Mycobacterium tuberculosis that most often affects the lungs. TB is curable and preventable. TB is spread from person to person through the air. When people with lung TB cough, sneeze or spit, they propel the TB germs into the air. A person needs to inhale only a few of these germs to become infected. About one-third of the world population has latent TB, which means people have been infected by TB bacteria but are not (yet) ill with the disease and cannot transmit the disease (Nkague, Timothee, Agouanent, & Ruther, 2019). According to world health organization 2018, Tuberculosis (TB) is one of the top 10 causes of death worldwide. In 2017, 10

million people fell ill with TB, and 1.6 million died from the disease (including 0.3 million among people with HIV), TB is a leading killer of HIV-positive people. In 2017, an estimated 1 million children became ill with TB and 230 000 children died of TB (including children with HIV associated TB). Multidrug-resistant TB (MDR-TB) remains a public health crisis and a health security threat. World Health Organization (WHO) estimates that there were 558 000 new cases with resistance to rifampicin – the most effective first-line drug, of which 82% had MDR-TB. Globally, TB incidence is falling at about 2% per year. This needs to accelerate to a 4–5% annual decline to reach the 2020 milestones of the "End TB Strategy". An estimated 54 million lives were saved through TB diagnosis and treatment between 2000 and 2017.

Ending the TB epidemic by 2030 is among the health targets of the Sustainable Development Goals (WHO, 2018). In the eighteenth century, Western Europe suffered terribly from this disease with a prevalence as high as 900 deaths per 100,000. This was largely due to poor ventilation, overcrowded housing, primitive sanitation, malnutrition among other risk factors that led to the epidemic. Today, this disease ranks as the second leading cause of morbidity and mortality in the world from a single infectious agent, after the human immunodeficiency virus (HIV) (Ronoh *et al.*, 2016). In 2017, nearly 2.5 million people who contracted TB lived in sub-Saharan Africa, and 665,000 of them died from the disease. On this day, we think of those who have died and we push for enriched efforts to keep more people from dying or even from becoming infected (WHO, 2019). Hippner *et al.* (2019) study showed that South Africa has the highest tuberculosis (TB) incidence rate in the world, and the disease is the leading cause of death in provinces with the rate of microbiologically confirmed TB (in 2012) ranging from approximately 400/100,000/year in Limpopo Province to almost 1,200/100,000/year in KwaZulu-Natal. In addition, the drivers of the epidemic also differ between provinces due to variation in factors including HIV prevalence, levels of drug resistance, and the capacity of the health system to identify and treat TB. Recent modeling studies have shown that scaling up existing interventions in all of South Africa has the potential to greatly reduce TB incidence and mortality, and to be cost-effective. However, the vast majorities of modelling studies have focused on the national level and have not addressed potential differences between provinces. Nigeria is among the 14 high burden countries for TB, TB/HIV and Multi Drug Resistant TB. The country is ranked seventh among the 30 high TB burden countries and second in Africa. The problem of TB in Nigeria has been made worse by the issues of drug resistant TB and the HIV/AIDS epidemic (WHO, 2019). It is estimated that 407,000 people in Nigeria have TB each year. In addition, there are an estimated 63,000 HIV positive people that get TB each year. An estimated 115,000 HIV negative people die from TB in Nigeria each year and an estimated 39,000 HIV positive people also die. It is difficult to appreciate what it means for 154,000

people to be dying each year from TB so it can be helpful to read the page on dying from TB (WHO, 2018). There are two forms of the disease namely: Latent TB and Active TB.

Latent TB: Latent tuberculosis infection (LTBI) is a state of persistent immune response to stimulation by Mycobacterium tuberculosis antigens without evidence of clinically manifested active TB. Someone has latent TB if they are infected with the TB bacteria but do not have signs of active TB disease and do not feel ill. However, they can develop active TB disease in the future. You have the germs in your body, but your immune system stops them from spreading. That means you do not have any symptoms and you are not contagious (Kiazyk & Ball, 2017). **ii Active TB disease:** Active tuberculosis is a condition in which the body's immune system is unable to fight off or defend against the mycobacterium tuberculosis bacterium. This means the germs multiply and can make one sick. One can spread the disease to others. Ninety percent of adult cases of active TB are from the reactivation of a latent TB infection (Okuonghae, 2013).

2.0 MATHEMATICAL FORMULATION

In this paper, we divide the population $N(t)$ into 5 compartments namely: $V(t)$, the number of TB-vaccinated individuals (*Bacillus Calmette Guerin* BCG); $S(t)$, the number of susceptible individuals; $L_T(t)$, TB-latently infected individuals who have no symptoms of TB disease and are not infectious; $I_T(t)$, TB-infected individuals, who have active TB disease and are infectious; $R_T(t)$, TB-recovered individuals; The recruitment of individual into vaccinated class is at a rate $\rho\Lambda$, where ρ the proportion of newborns that are vaccinated and Λ is the birth rate. Here we assume that the waning BCG vaccine moves individuals to the susceptible compartment at a rate ν . The recruitment into susceptible compartment is at a rate $(1-\rho)\Lambda$ where $1-\rho$ is the proportion of immigrant and new born that are not vaccinated at birth. The susceptible compartment decreases due to natural

death rate μ . Susceptible individuals acquire TB infection from the expression $f\lambda_T\beta cS$, where $\lambda = \frac{I_T}{N}$, β is the effective contact rate for TB infection, f is the proportion of individuals that develop slow TB strain, c is the contact rate per unit time and by the function $(1-f)\lambda_T\beta cS$, where $(1-f)$ is the proportion of individuals that develop fast TB strain. Latently infected TB compartment increases by incidences $f\lambda_T\beta cS$ and $f_1\delta_2\beta c\lambda_T R_T$, where f_1 is the proportion of recovered TB that relapse to latently infected TB. The latently infected TB individuals decreases due to κL_T where κ is the endogenous reactivation and by incidence $\delta_1\beta c\lambda_T L_T$ where δ_1 exogenous re-infection rate and by natural death at a rate μ . Active TB compartments increases by κL_T and incidences $(1-f)\beta c\lambda_T S$, $f_1\delta_1\beta c\lambda_T L_T$ and $(1-f_1)\delta_2\beta c\lambda_T$ where $1-f_1$ is the proportion that relapse back into active TB. TB induce death eI_T where e is TB induced death rate and μI_T where μ is the natural death rate in the entire compartment. The recovered TB compartment increases by τI_T and decreases by incidences $f_1\delta_1\beta c\lambda_T R_T$ and by μR_T .

The epidemiological flow for the Tuberculosis transmission model is presented in figure 1.

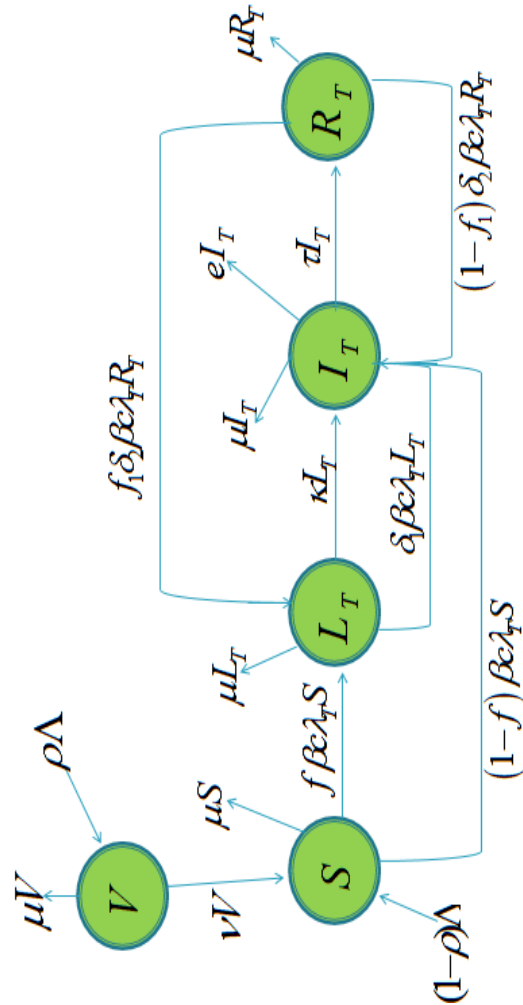


Figure 1. The epidemiological flow for the Tuberculosis transmission model

2.1 Model Equations

$$\left. \begin{aligned}
 \frac{dV}{dt} &= \rho\Lambda - (\mu + V)V \\
 \frac{dS}{dt} &= vV + (1 - \rho)\Lambda - \frac{f\beta cI_T}{N}S \\
 &\quad - \frac{(1-f)\beta cI_T}{N}S - \mu S \\
 \frac{dL_T}{dt} &= \frac{f\beta cI_T}{N}S + \frac{f_1\delta_2\beta cI_T}{N}R_T \\
 &\quad - \frac{\delta_1\beta cI_T L_T}{N} - \mu L_T - \kappa L_T \\
 \frac{dI_T}{dt} &= \kappa L_T + (1-f)\frac{\beta cI_T}{N}S + \frac{\delta_1\beta cI_T L_T}{N} \\
 &\quad + (1-f_1)\frac{\beta c\delta_2 I_T R_T}{N} - (e + \mu + \tau)I_T \\
 \frac{dR_T}{dt} &= \mu_3\tau I_T - \frac{f_1\delta_2\beta cI_T R_T}{N} \\
 &\quad - \frac{(1-f_1)\delta_2\beta cI_T R_T}{N} - \mu R_T
 \end{aligned} \right\} \quad (1)$$

with initial conditions
 $V(0) = V_0, S(0) = S_0, L_T(0) = L_{T0},$
 $I_T(0) = I_{T0}, R_T(0) = R_{T0}$
 $N(0) = N_0$
 $\lambda_T = \frac{I_T}{N}$ (2)

Table 1 Definition of parameters and Variable for TB Model

Parameters/Variable	Description
f	Proportion of individuals that develop slow TB strain.
f_1	Proportion of recovered TB that relapses to latently infected TB.
c	Contact rate per unit time.
$(1-f)$	Proportion of individuals that develop fast TB strain.
ρ	The proportion of newborns that are vaccinated at birth.
$1-\rho$	Proportion of newborns that are not vaccinated at birth.
$1-f_1$	Proportion that relapse back into active TB.
κ	Endogenous reactivation.
Λ	Birth rate.
β	The effective contact rate for TB infection.

- μ The natural death rate in each compartment.
- v The waning BCG vaccine.
- e_1 Active TB induced death.
- δ_2 Probability of relapsed into Latent and active TB.
- τ Recovery rate.
- δ_1 Exogenous re-infection rate.
- $V(t)$ Population of vaccinated TB at time t .
- $S(t)$ Population of susceptible at time t .
- $L_T(t)$ Population with latent TB at time t .
- $I_T(t)$ Population with active TB at time t .
- $R_T(t)$ Population of TB-recovered class at time t .

3.0 MODEL ANALYSIS

3.1 Invariant region

Let $\Omega = \{V, S, L_T, I_T, R_T\} \in R_+^5$ be any solution of the system (1) with non-negative initial conditions and applying Birkhoff & Rota (1982) theorem on differential equality we obtain

$0 \leq N \leq \frac{\Lambda}{\mu}$ as $t \rightarrow \infty$. Therefore, the feasible

solution set of the equations (1) enter the region,

$$\left\{ \Omega = (V, S, L_T, I_T, R_T) = R_+^5 / V > 0, S > 0, L_T \geq 0, I_T \geq 0, R_T \geq 0, N_T \leq \frac{\Lambda}{\mu} \right\} \quad (3)$$

3.2 Positivity of Solution for TB Model

It is necessary to prove that all solutions of system (1) with positive initial conditions will remain positive for all times $t > 0$. Let the initial data be

$$\{V(0), S(0), L_T(0), I_T(0), R_T(0) \geq 0\} \in \mathbb{R}_+^5.$$

Then, the solution set $\{V(t), S(t), L_T(t), I_T(t), R_T(t)\}$ of the system (1) is positive for all $t > 0$.

3.3 Disease Free and The Endemic Equilibrium

setting $\frac{dV}{dt} = \frac{dS}{dt} = \frac{dL_T}{dt} = \frac{dI_T}{dt} = \frac{dR_T}{dt} = 0$ the disease free equilibrium state (DFE) and (EES) denoted by E_0 and E_1 of system (1) is given by

$$E_0(T) = \left(V_T^0, S_T^0, L_T^0, I_T^0, R_T^0 \right) = \left\{ \frac{\rho\Lambda}{(\mu+\nu)}, \frac{(\Lambda\nu+(1-\rho)\Lambda\mu)}{(\mu+\nu)}, 0, 0, 0 \right\} \quad (3)$$

$$\left. \begin{aligned} V^* &= \frac{\rho\Lambda}{(\mu+\nu)} \\ S^* &= \frac{\Lambda(\mu(1-\rho)+\nu)}{(\beta c\lambda_T^* + \mu)(\mu+\nu)} \\ E_1 = L_T^* &= \frac{(fS^* + f_1\delta_2 R_T^*)\beta c\lambda_T^*}{(\delta_1\beta c\lambda_T^* + \mu + \kappa)} \\ I_T^* &= \frac{\kappa L_T^* (\beta c\lambda_T^* + \mu)(\mu+\nu) + (1-f)\beta c\lambda_T^* (\Lambda(\mu(1-\rho)+\nu))}{(1-f_1)\beta c\delta_2\lambda_T^*\tau - (e_1 + \mu + \tau)(\delta_2\beta c\lambda_T^* + \mu)} \\ R_T^* &= \frac{\tau I_T^*}{(\delta_2\beta c\lambda_T^* + \mu)} \end{aligned} \right\} \quad (4)$$

3.4 Model Reproduction Number

The basic reproduction number of Tb disease transmission dynamics is denoted by R_0 . It is a critical parameter used to investigate the behavior of epidemiological models. It is defined as the average number of secondary infections infected by an infective individual during an infective period if the entire population is susceptible. We use Diekman and Heesterbeek's (1990) next generation matrix technique to calculate the basic reproduction number for Tb Disease by taking into account the infected compartments of the system (1). Let F_i be the rate of appearance of new infection in the i compartment and V_i be the rate of transfer of individuals out of i , given the disease free equilibrium, then R_0 is the spectral radius (largest Eigen values) of the next generation matrix denoted by $G = FV^{-1}$.

$$R_0 = \rho FV^{-1}$$

$$R_{0T} = \frac{c\beta\mu^2(\nu + \Lambda\mu - \Lambda\mu\rho)(\kappa + \mu - 2f\kappa - f\mu)}{(\tau + \mu + e_1)(\kappa + \mu)(\nu + \mu)} \quad (5)$$

3.5 Testing of Bifurcation for TB

To solve for λ_T^* we substitute I_T^* in equation (4) into equation (2) to get

$$\lambda_T^* = \frac{\kappa(f\beta c\lambda_T^* S^* + f_1\delta_2\beta c\lambda_T^* R_T^*)(\beta c\lambda_T^* + \mu)(\mu+\nu)\mu + (1-f)\beta c\lambda_T^*(\delta_1\beta c\lambda_T^* + \mu + \kappa)(\Lambda(\mu(1-\rho)+\nu))\mu}{((1-f_1)\beta c\delta_2\lambda_T^*\tau - (e_1 + \mu + \tau)(\delta_2\beta c\lambda_T^* + \mu)(\delta_1\beta c\lambda_T^* + \mu + \kappa))\Lambda} \quad (6)$$

Evaluating (6) we have

$$\lambda_T^* (A_1\lambda_T^{*2} + A_2\lambda_T^* + A_3) = 0 \quad (7)$$

where

$$\begin{aligned} A_1 &= c^2\Lambda\beta^2\delta_1\delta_2(2+\tau) \\ A_2 &= c\Lambda\beta\mu^2\delta_2 - c\Lambda\beta\mu^2\delta_1 + e_1\kappa\Lambda\beta\delta_2 \\ &\quad + c\kappa\Lambda\beta\tau\delta_2 + c\Lambda\beta\tau\mu\delta_1 + c\Lambda\beta\tau\mu\delta_2 \\ &\quad + c\Lambda\beta\tau\delta_2\mu^2f + c\Lambda\beta\tau\delta_2 - c^2\beta^2\mu^2\delta_1 \\ &\quad + c^2\Lambda\beta^2\mu^2\rho\delta_1 - c^2f\kappa\beta^2\mu^2S^* \\ &\quad - c^2\Lambda\beta^2\mu\nu\delta_1 + c^2f\Lambda\beta^2\mu^2\delta_1 \\ &\quad - c^2f\Lambda\beta^2\mu^2\rho\delta_1 - c^2\kappa\beta^2\delta_2\mu^2f_1R_T \\ &\quad + c^2f\Lambda\beta^2\mu\nu\delta_1 - c^2f\kappa\beta^2\mu\nu S^* \\ &\quad - c^2\kappa f_1\beta^2\mu\nu\delta_2R_T \\ A_3 &= \Lambda\mu^2e_1 + \kappa\Lambda\mu^2 + \Lambda\tau\mu^2 + e_1\kappa\Lambda\mu + \kappa\Lambda\tau\mu + \Lambda\mu^3 \\ &\quad - cf\kappa\beta\mu^2\nu S^* - c\Lambda\beta\mu^3 + cf\beta\Lambda\mu^3 - c\kappa\Lambda\beta\mu^2 \\ &\quad - c\beta\Lambda\mu^2\nu + c\Lambda\beta\mu^3\rho - \kappa c\Lambda\beta\mu\nu\delta_1 - f\kappa\Lambda\beta\mu^2 \\ &\quad + cf\Lambda\beta\mu^2\nu - f\kappa\Lambda\beta\mu^3\rho + c\kappa\Lambda\beta\mu^2\rho - cf\kappa\beta\mu^3S^* \\ &\quad - c\kappa\beta\mu^2\delta_2f_1R_T + cf\kappa\Lambda\beta\mu^2\rho\nu - cf\kappa\Lambda\beta\mu^2\rho \\ &\quad - c\kappa f_1\beta\mu^2\nu\delta_2R_T \end{aligned}$$

CASE 1: $\lambda_T^* = 0$ corresponds to Disease Free

Equilibrium point (DFE) whose stability was established.

CASE 2: $A_1\lambda_T^{*2} + A_2\lambda_T^* + A_3 = 0$ $\lambda_T^* > 0$

corresponds to a situation when the disease persists (endemic). In case of backward bifurcation, multiple endemic equilibria must exist. This implies that equation (4.41) indicates three cases which we have to consider if $\lambda_T^* = 0$ depending on the sign of A_2 and A_3 since A_1 is always positive (Gakii & Malonza, 2018).

1 If $A_2 < 0$ and $A_3 = 0$ or $A_2^2 - 4A_1A_3 > 0$ then equation (6) has unique endemic equilibrium point (one positive root) and no backward bifurcation possibility.

2 If $A_3 > 0$, $A_2 > 0$ and $A_2^2 - 4A_1A_3 > 0$ then equation (6) has two endemic equilibrium points (two positive roots), and therefore it is possible for backward bifurcation to occur.

3. Otherwise, there is none.

However, it is important to note that A_3 is always positive if $R_{0T} < 1$ and negative if $R_{0T} > 1$. Hence the above examination leads to the following theorem:

Theorem 1: The TB model has the following

- i. Precisely one unique endemic if $A_3 < 0 \Leftrightarrow R_{0T} > 1$.
- ii. Precisely one unique endemic equilibrium point if $A_2 < 0$ and $A_3 = 0$, $A_2 - 4A_1A_3 = 0$
- iii. None if otherwise

The Endemic Equilibrium Point (EEP) exist if and only if $R_{0T} > 1$

Proof:

Since A_1 is always positive, that is $A_1 > 0$, then

$$\lambda_T^* = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$$

this implies that

$\lambda_T^* = \text{negative}$ or $\lambda_T^* = \text{positive}$ value, by applying the properties of quadratic equation. However, we cannot have $\lambda_T^* = \text{negative}$ because we cannot have the rate infection as negative epidemiologically, therefore $\lambda_T^* > 0$ provided

$A_3 > 0$. From the theorem 1, (i), the last condition is equal to $R_{0T} > 1$. From (iii) backward bifurcation is possible if we set the discriminant $A_2^2 - 4A_1A_3$ and solving for the critical value of $R_{0T}^c > 1$ we get

$$R_{0T}^c = 1 - \frac{A_2^2}{4[1 + \mu + \tau]c^2\Lambda\beta^2\delta_1\delta_2A_3} \tag{8}$$

From equation (8) it can be shown that the backward bifurcation occur for the value of R_{0T} lying in the range of $R_{0T}^c < R_{0T} < 1$. Therefore, if $R_{0T}^c < R_{0T} < 1$ the model exhibits backward bifurcation. Thus, the endemic equilibrium point EEP exists, if and only if $R_{0T} > 1$. Thus, end of the proof. Therefore the TB exhibits a backward bifurcation which occur when the endemic equilibrium point EEP exist if and only if $R_{0T} < 1$ (Namawejeje, 2011).

3.6 Sensitivity Analysis for TB

In particular, sensitivity indices of the basic reproduction number, R_{0T} with respect to the model parameter are examined. From equation (5) we compute the sensitivity indices using the formula

$$r_\theta^{R_{0T}} = \frac{dR_{0T}}{d\theta} \times \frac{\theta}{R_{0T}}$$

and the result is shown in the

table 2 below

Table: 2 signs of sensitivity indices for R_{0T}

Parameter	Sensitivity index
f	1
v	positive
κ	positive
e	1
τ	negative
ρ	negative
c	1
μ	positive

Sensitivity analysis of R_{0T} with respect to the model parameters were carried out. The positive ones show that, the waning of BCG vaccine,

endogenous reactivation, active TB induced death, contact rate per unit time and natural death rate in each compartment are highly sensitive parameters that affect the transmission dynamics of TB in any population. Thus, vaccination is a good strategy that can lead to controlling TB transmission (Adulrahman, Ifeoluwa, Bamidele, & Yusuf, 2010).

3.7 Stability Analysis Using Bifurcation Method for TB

The centre manifold theory can be used to analyze the stability near the DFE (E_0) and $R_{0T} = 1$. Therefore, from equation (5) we let $c\beta = \theta$ be the bifurcation parameter and $R_{0T} = 1$ be the bifurcation point and making θ subject of the formula we have

$$\theta = \frac{(\mu + \nu)(\mu + \kappa)(e_1 + \mu + \tau)}{f\mu^2(2\kappa + \mu)(\mu + \nu - \mu\rho)}$$

also, let

$$x_1 = V, x_2 = S, x_3 = L_T, x_4 = I_T, x_5 = R_T \quad (9)$$

Substituting (9) into equations (1) we have

$$\left. \begin{aligned} \dot{x}_1 &= f_1 = \rho\Lambda - (\mu + \nu)x_1 \\ \dot{x}_2 &= f_2 = \nu x_1 + (1 - \rho)\Lambda - \frac{\theta x_4 x_2}{N} - \mu x_2 \\ \dot{x}_3 &= f_3 = \frac{f\theta x_4 x_2}{N} + \frac{f_1 \delta_2 \theta x_4 x_5}{N} - (\mu + \kappa)x_3 - \frac{\delta_1 \theta x_4 x_3}{N} \\ \dot{x}_4 &= f_4 = \kappa x_3 + \frac{(1 - f)x_4 x_2}{N} + \frac{\delta_1 \theta x_4 x_3}{N} + \frac{(1 - f_1)\theta \delta_2 x_4 x_5}{N} - (e_1 + \mu + \tau)x_4 \\ \dot{x}_5 &= f_5 = \tau x_4 - \mu x_5 - \frac{f_1 \delta_2 \theta x_4 x_5}{N} - \frac{(1 - f_1)\delta_2 \theta x_4 x_5}{N} \end{aligned} \right\} 10$$

Differentiating (10) at disease free equilibrium we have

$$\begin{bmatrix} -(\mu + \nu) & 0 & 0 & 0 & 0 \\ \nu & -\mu & 0 & a_1 & 0 \\ 0 & 0 & -(\mu + \kappa) & a_2 & 0 \\ 0 & 0 & \kappa & a_3 & 0 \\ 0 & 0 & 0 & \tau & -\mu \end{bmatrix} \quad (11)$$

$$a_1 = \frac{\theta(\Lambda\nu + \Lambda\mu - \rho\Lambda\mu)\mu}{N(\mu + \nu)}, a_2 = \frac{f\theta(\Lambda\nu + \Lambda\mu - \rho\Lambda\mu)\mu}{N(\mu + \nu)}$$

$$a_3 = \frac{\theta(1 - f)(\Lambda\nu + \Lambda\mu - \rho\Lambda\mu)\mu}{N(\mu + \nu)} - (e_1 + \mu + \tau)$$

Evaluating (11) for right and left eigenvalues we

$$\left. \begin{aligned} W_1 &= 0 \\ W_2 &= -\frac{\theta(\Lambda\nu + \Lambda\mu - \rho\Lambda\mu)}{N(\mu + \nu)}W_4 \\ \text{have } W_3 &= \frac{f\theta(\Lambda\nu + \Lambda\mu - \rho\Lambda\mu)\mu}{N(\mu + \nu)(\mu + \kappa)}W_4 \\ W_4 &= w_4 \text{ free} \\ W_5 &= \frac{\tau W_4}{\mu} \end{aligned} \right\} (12)$$

$$\left. \begin{aligned} V_1 &= 0 \\ V_2 &= 0 \\ V_3 &= \frac{\kappa V_4}{(\mu + \nu)} \\ V_4 &= V_4 \text{ free} \\ V_5 &= 0 \end{aligned} \right\} (13)$$

After computing the right and left eigenvalues we have to establish the conditions for the existence of backward bifurcation by determining the sign of **a** and **b** as indicated in Theorem below.

Theorem 2. Consider the following general system of ordinary differential equations with a parameter θ .

$$\frac{dx}{dt} = f(x, \theta), f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$$

and $f \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R})$ (14)

where θ is an equilibrium point of the system, that is, $f(0, \theta) \equiv 0 \forall \theta$ and

i. $A = D_x f(0, 0) = \left[\frac{\partial f_i}{\partial x_i}(0, 0) \right]$ is the

linearization matrix of the system around the equilibrium with θ evaluated at 0.

ii. Zero is a simple eigenvalue of A and all other eigenvalue A have negative real parts.

iii. Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalues.

Let f_k be the K th component of f and

$$a = \sum_{k,i,j=1}^n v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0) \quad (15)$$

and

$$b = \sum_{k,i=1}^n v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \theta}(0, 0) \quad (16)$$

The local dynamics of system (10) around equilibrium are totally governed by the signs of

a and b . Thus, the following conditions:

- i. $a > 0, b > 0$, when $\theta < 0$, with $|\theta| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \theta \ll 1$, The equilibrium is unstable and there exists a negative and locally asymptotically stable equilibrium.
- ii. $a < 0, b < 0$ when $\theta < 0$ with $|\theta| \ll 1$, 0 is unstable; when $0 < \theta \ll 1$, the equilibrium is locally asymptotically stable, and there exists a positive unstable equilibrium.
- iii. $a > 0, b < 0$, when $\theta < 0$, with $|\theta| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \theta \ll 1$, 0 is stable, and a positive unstable equilibrium appears.
- iv. $a < 0, b > 0$, when θ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable. Particularly if

$a > 0$, and $b > 0$, then a backward bifurcation occurs at $\theta = 0$: (Olaniyi, Lawal, & obabiya, 2016).

Computation of a and b

Since $V_1 = V_2 = V_5 = 0$ For $K = 1, \dots, 5$, we can only consider $K = 3, 4$. The only partial derivatives different from zero are:

$$\left. \begin{aligned} \frac{\partial^2 f_3}{\partial x_3 \partial x_4} &= -\frac{\delta_1 \theta}{N}, \frac{\partial^2 f_3}{\partial x_4 \partial x_2} \\ &= \frac{f \theta}{N}, \frac{\partial^2 f_3}{\partial x_4 \partial x_5} = \frac{f_1 \delta_2 \theta}{N} \\ \frac{\partial^2 f_4}{\partial x_4 \partial x_2} &= \frac{(1-f)\theta}{N}, \frac{\partial^2 f_4}{\partial x_4 \partial x_3} \\ &= \frac{\delta_1 \theta}{N}, \frac{\partial^2 f_4}{\partial x_4 \partial x_5} = \frac{(1-f_1)\delta_2}{N} \end{aligned} \right\} \quad (17)$$

Substituting equations (12), (13) and (17) into equation (16) and evaluating the results we have

$$a = \left. \begin{aligned} &\frac{\kappa \theta^2 \delta_1 (v + \mu - \rho \mu) \mu^2}{(\mu + v)^2} V_4 W_4^2 \\ &+ \frac{\kappa f^2 \theta^2 (v + \mu - \rho \mu) \mu^2}{(\mu + v)^2 (\mu + \kappa)} V_4 W_4^2 \\ &+ \frac{\kappa f_1 \delta_2 \theta \tau \mu}{\Lambda \mu (\mu + v)} V_4 W_4^2 - \\ &\frac{\theta^2 (1-f)(v + \mu - \rho \mu) \mu^2}{(\mu + v)} V_4 W_4^2 + \\ &\frac{f \delta_1 \theta^2 (v + \mu - \rho \mu) \mu^2}{(\mu + v)(\mu + \kappa)} V_4 W_4^2 \\ &+ \frac{(1-f_1) \mu \delta_2 \tau}{\Lambda \mu} V_4 W_4^2 \end{aligned} \right\} \quad (18)$$

Evaluating equation we have (10)

$$\left. \begin{aligned} \frac{\partial^2 f_3}{\partial x_4 \partial \theta} &= \frac{f(v + \mu - \rho \mu) \mu}{(\mu + v)} \\ \frac{\partial^2 f_4}{\partial x_4 \partial \theta} &= \frac{(1-f)(v + \mu - \rho \mu) \mu}{(\mu + v)} \end{aligned} \right\} \quad (19)$$

Substituting equation (12), (13), and (19) into equation (15) and evaluating the result we have

$$b = \frac{(\nu + \mu - \rho\mu)\mu}{(\mu + \nu)} (fV_3W_4 + (1-f)V_4W_4) \quad (20)$$

Since $a > 0$ and $b > 0$, the first condition in theorem 2 holds. Thus, the TB model exhibits a backward bifurcation.

3.8 Hopf Bifurcation Analysis for TB Model

The equation (10) has a Hopf-Bifurcation around the endemic equilibrium point E_T^* satisfying the condition in equation (21).

$$\beta = \bar{\beta} = \frac{5\mu + \kappa + \nu + e_1 + \tau}{\left(\frac{\delta_1 c I_T}{N} + \frac{c I_T}{N} + \frac{(1-f)cS}{N} + \frac{\delta_1 c L_T}{N} + (1-f_1) \frac{\delta_1 c R_T}{N} + \frac{\delta_2 \beta c I_T}{N} \right)} \quad (21)$$

Now the Jacobian matrix of equation (10) at endemic equilibrium point is given as

$$\begin{bmatrix} -A & 0 & 0 & 0 & 0 \\ \Lambda & -B & 0 & -C & 0 \\ 0 & D & -E & F & G \\ 0 & H & I & J & K \\ 0 & 0 & 0 & L & -M \end{bmatrix} \quad (22)$$

where

$$\left. \begin{aligned} A &= (\mu + \nu), B = \left(\frac{\beta c I_T}{N} + \mu \right), C = \frac{\beta c S}{N}, \\ D &= \frac{f \beta c I_T}{N}, E = \left(\mu + \kappa + \frac{\delta_1 \beta c}{N} \right) \\ F &= \frac{f \beta c S}{N} + \frac{f_1 \delta_2 \beta c R_T}{N} - \frac{\delta_1 \beta c L_T}{N}, \\ G &= \frac{f_1 \delta_2 \beta c I_T}{N}, H = \frac{(1-f) \beta c I_T}{N}, \\ I &= \frac{\kappa + \delta_1 \beta c I_T}{N}, J = \frac{(1-f) \beta c S}{N} + \frac{\delta_1 \beta c L_T}{N} \\ &+ \frac{(1-f_1) \beta c \delta_1 R_T}{N} - (e_1 + \mu + \tau), \\ K &= \frac{(1-f_1) \beta c \delta_2 I_T}{N}, L = \tau - \frac{\delta_2 \beta c R_T}{N}, \\ M &= \left(\mu + \frac{\delta_2 \beta c I_T}{N} \right) \end{aligned} \right\} \quad (23)$$

By Solving equation (22) using scientific work place 5.5 the system has one negative eigenvalue $-(\mu + \nu)$ and the remaining four eigenvalues are complex conjugate.

Clearly, the condition for the Hopf-Bifurcation is satisfied as $\beta = \bar{\beta}$. Now, we check the second condition in the following $\frac{dT}{d\beta} \neq 0$. From equation

(22) finding the trace of the matrices (T) and differentiating same with respect to β we have

$$\frac{dT}{d\beta} = \frac{(1-f)cS}{N} + \frac{\delta_1 \beta c L_T}{N} + \frac{(1-f_1) \beta c \delta_1 R_T}{N} \neq 0 \quad (24)$$

Hence, the system at the parameter $\beta = \bar{\beta}$ has a Hopf-Bifurcation around the endemic equilibrium which means that the susceptible, latent and recovery have to be controlled or else an endemic will occur (See: Ali, 2015).

4.0 CONCLUSION

The TB model is both biologically and mathematically feasible. In addition, we obtained disease-free and endemic equilibrium. The system's reproduction number is determined. Before performing stability analysis using the bifurcation method, we first tested for bifurcation and the findings demonstrate $\lambda_T^* = 0$ which indicate disease-free and the quadratic equation were analyzed and we conclude that backward bifurcation is possible using critical value. The model is subjected to sensitivity analysis to determine the parameters that influence reproduction. The model's stability is examined using the Bifurcation method, and the results of transcriptional bifurcation indicate that the model is locally asymptotically stable because a both b and are greater than zero. Hopf-Bifurcation around the endemic equilibrium was obtained, and the results show that susceptible, latent, and recovery must be controlled or an endemic will occur.

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