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Optimal Investment Problem in a Financial Institution: The Effect of Risk Aversion and Market Price of Risk Parameters on Optimal Investment

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ABSTRACT

This research work looked at how to optimally allocate the total wealth of a financial institution in asset portfolio that is made up of three assets, which are treasury, security and loan. In the financial institution's portfolio optimization problem, the interest rate is stochastic, the volatility of the security price is assumed to be described by the model of Heston stochastic volatility. By applying stochastic optimization theory, we obtained the optimum investment policy or strategy for the financial institution for the case of utility function which is constant relative risk aversion (CRRA). Numerical examples were also given to illustrate the dynamics of the optimum investment policy.

Keywords: financial institution; investment policy; stochastic optimization theory; stochastic interest rate and stochastic volatility.

1.0 INTRODUCTION

Optimal Assets allocation plays a vital role in banks and other financial institutions. One major issue in optimal assets allocation problems is how the optimal rebalancing pattern of assets through time can be characterized (Peter et al., 2011). Optimization of the terminal wealth expected utility is one method of dealing with the above problem.

Optimal assets allocation and capital management also play a very important role in financial institutions. In recent time, interest in this topic in stochastic framework has grown commensurately (Peter *et al.*, 2011; Fatma and Fathi, 2014; Grant and Peter, 2014). Peter *et al*. (2011), studied an optimization problem with stochastic interest rates which considered some characteristics of bank. Their aim is to illustrate the numerical aspect of the derived equation which is Hamilton – Jacobi – Bellman (HJB) and to focus on the problem's results from practical perspective. Danjuma *et al.* (2019), also considered assets allocation problem. In their work, they illustrated that analytic approach can be employed in solving the asset allocation problems. They formulated an optimal asset allocation problem in stochastic interest rate and constant volatility framework and obtained the optimal investment strategy through the application of the stochastic optimization theory for the case of constant relative risk aversion (CRRA) utility function.

Many papers have studied how financial institutions can optimally allocate its total wealth among its assets by applying stochastic control theory in discrete and continuous time setting (Van Schalkwyk and Witbooi, 2017; Danjuma and Dange, 2022). The approach solved partial differential HJB equation which is nonlinear and obtained the value function solution in closed form.

The paper by Mukudden – Peterson and Peterson (2006), studied the optimum risk management problem of banks in a framework that is stochastic. In their research, they optimized market and capital adequacy risk including held assets safety and capital sources stability respectively. Obtained the solution of the optimization problem by applying dynamic programming principle. Also in another work by Mukudden – Peterson and Peterson, (2008), determined how additional debt and equity optimal rate can be raised. It also determined the bank's equity optimal allocation policy. They employed stochastic optimization theory to verify their results. Furthermore, in the work of Mukudden – Peterson *et al.* (2007), they obtained the HJB equation solution derived from the formulated optimization problem analytically for the case of power, exponential and logarithmic utility functions. Here, depository consumption, quantity of the depository invested in loans and loan losses provisions are the control variates.

Mulaudzi *et al.* (2008) studied how the funds of a financial institution can be optimally allocated in treasuries, securities under risk and regret theoretical setting. In Fouche *et al.* (2006), they derived the leverage, equity and Tier 1 ratios continuous time stochastic models and capital adequacy ratio (CAR). They also show the relevant of their result to the banking sector by deriving the leverage ratio optimal investment policy on a given time internal. Precisely, they obtained the leverage ratio optimal expected utility and also derived the optimal investment policy that can make it likely to optimize the leverage ratio expected terminal utility for a given interval of time.

Current literature has largely focused on financial institutions optimal asset allocation in stochastic interest framework. However, in a more realistic world, one should account for both stochastic interest rates and stochastic volatility (Danjuma et al., 2020; Hui and Yonggan, 2013). This research attempts to study a financial institution's asset allocation problem, effect of risk aversion and market price of risk parameters on the optimum investment policy in the setting where the interest rate and volatility are stochastic.

2.0 MATHEMATICAL MODELS FORMULATION

We consider a financial institution that dynamically allocates its total wealth among three assets namely: treasury, loan and security. The assets prices satisfy the geometric Brownian motion, assets can be acquired and traded with no transaction charges or constraint on short sales. The interest rate is stochastic and the risk preference of the investor satisfies CRRA utility function.

2.1 The Financial Assets Models

Here, we take the financial market to be complete, frictionless and continuously open over a fixed time interval $[0,T]$. The Brownian motion is defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ which is complete; the generated filtration by the Brownian motions is $\mathcal{F}=\{\mathcal{F}_t\}_{t\geq 0}$, real world probability is $\mathbb{P}$ and the sample space is Ω. In the financial market, we take the riskless treasury as the first asset. The following stochastic differential equation describes asset price dynamics:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \qquad S(0)=S_0 \qquad (1)$$

The dynamics of the interest rate $r(t)$, can also be described by the stochastic model

$$dr(t) = \big(a - br(t)\big)dt \\ -\sigma_r\sqrt{r(t)}dw_r(t), \quad r(t)=r_0 \qquad (2)$$

where a, b and $\sigma_r = \sqrt{k_1}$ are constants.

Another asset is a loan, whose price at time $t \geq 0$ can be given by $L(t)$. The stochastic loan model is given by equation (3) below:

$$\frac{dL(t)}{L(t)} = \big(r(t) + b_1\lambda_r k_1 r(t)\big)dt \\ +b_1\sigma_r\sqrt{r(t)}dw_r(t) \qquad (3)$$

where b_1, λ_r and k_1 are constants. The risk premium $b_1\lambda_r r(t)$ of loan return changes with t both over the dependency on $r(t)$ implicitly and over the dependency on b_1 explicitly.

Risky security is the third asset in the financial market. The stochastic security model at $t \geq 0$, is given by:

$$\frac{dS(t)}{S(t)} = \big(r(t) + \upsilon\eta(t) + \sigma_s\lambda_r k_1 r(t)\big)dt \\ +\sigma_s\sigma_r\sqrt{r(t)}dw_r(t) \\ +\sqrt{\eta(t)}dw_s(t) \qquad (4)$$

The volatility $\eta(t)$ is assumed to satisfy the Heston model:

$$d\eta(t) = \alpha\big(\delta - \eta(t)\big)dt \\ +\sigma_\eta\sqrt{\eta(t)}dw_r(t) \qquad (5)$$

where α, δ and σ_η are positive constant and satisfied the condition $2\alpha\delta > \sigma_\eta^2$ and it ensures $\eta(t) > 0\ \forall\ t \in [0,T]$.

Here, we assume that there is no correlation between $w_s(t)$ and $w_r(t)$, and between $w_\eta(t)$ and $w_r(t)$. The correlation between $w_s(t)$ and $w_\eta(t)\ is\ \rho$.

2.2 The Asset Portfolio Model of the Financial Institution

We take $X(t)$ as the worth of financial institution portfolio at time $t \in [0,T]$, $\pi_s(t)$ the amount invested in the security and $\pi_l(t)$ the amount invested in loan respectively. Therefore, $\pi_0(t) = X(t) - \pi_s(t) - \pi_l(t)$ gives the amount put into the riskless asset. A model for the asset portfolio is:

$$dX(t) = \left(X(t) - \pi_s(t) - \pi_l(t)\right)\frac{dS_0(t)}{S_0(t)} + \pi_s(t)\frac{dS(t)}{S(t)} + \pi_l(t)\frac{dL(t)}{L(t)}$$
$$= [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]dt$$
$$+\left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right]dw_r(t) + \pi_s(t)\sqrt{\eta(t)}dw_s(t) \quad (6)$$

2.3 Admissible Policy

An admissible investment policy $\Pi(t) = (\pi_s(t), \pi_l(t))$ is the one that satisfies the following conditions:

i. $\pi_s(t)$ and $\pi_l(t)$ are all $f_t -$ measurable.

ii. $E\left(\int_0^T (\pi_s^2(t)\eta(t) + [\pi_s(t)\sigma_s\sigma_r + \pi_l(t)b_1\sigma_r]^2 r(t))dt\right) < \infty$

iii. There exist a unique solution of equation (6) $\forall\, \pi(t) = (\pi_s(t), \pi_l(t))$.

2.4 The Portfolio Optimization Problem

Let Π be the entire admissible strategy. Under a portfolio (6), the financial institution looks for an optimal strategy $\pi_s{}^*(t)$ and $\pi_l{}^*(t)$ which maximizes the terminal wealth expected utility. i.e.:

$$\max_{\pi(t)\in\Pi} E[U(X(T))] \quad (7)$$

Based on stochastic optimization theory, we state the problem as follows:

Maximize $E[U(X(T))]$

Subject to:

$$dr(t) = \left(a - br(t)\right)dt - \sigma_r\sqrt{r(t)}dw_r(t)$$
$$d\eta(t) = \alpha\left(\delta - \eta(t)\right)dt + \sigma_\eta\sqrt{\eta(t)}dw_r(t)$$
$$dX(t) = [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]dt + \left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right]dw_r(t) + \pi_s(t)\sqrt{\eta(t)}dw_s(t)$$
$$X(0) = x_0, r(0) = r_0, \eta(0) = \eta_0$$

where $0 \le t \le T$ and $X(0) = x_0$, $r(0) = r_0, \eta(0) = \eta_0$ are the initial conditions of the optimal control problem.

Here, we want to maximize a terminal wealth anticipated utility of a financial institution's portfolio at date $T > 0$ in future. That is, find an optimal value function

$$H(t,r,\eta,x) = \max_{\pi(t)\in\Pi} E\left[U\left(X(T)\right)\middle| r(t) = r, \eta(t) = \eta, X(t) = x\right] \quad (8)$$

and the optimum investment policy is $\pi^*(t) = (\pi_s{}^*(t), \pi_l{}^*(t))$ such that

$$\underset{\pi^*(t)}{\mathrm{H}}(t,r,\eta,x) = H(t,r,\eta,x) \quad (9)$$

2.5. The Transformation of the Portfolio Optimization Problem to HJB Equation

The portfolio optimization problem HJB equation is:

$$\max_{\pi(t)\in\Pi}\{H_t + [X(t)r(t) + \pi_s(t)\upsilon\eta(t) + \pi_s(t)\sigma_s\lambda_r k_1 r(t) + \pi_l(t)b_1\lambda_r k_1 r(t)]H_x$$
$$+\frac{1}{2}(\pi_s^2(t)\eta(t) + \left[\pi_s(t)\sigma_s\sigma_r\sqrt{r(t)} + \pi_l(t)b_1\sigma_r\sqrt{r(t)}\right]^2)H_{xx}$$
$$- [\pi_s(t)\sigma_s\sigma_r^2 r(t) + \pi_l(t)b_1\sigma_r^2 r(t)]H_{xr} + \left[\rho\pi_s(t)\sigma_\eta\eta(t)\right]H_{x\eta} + [a - br(t)]H_r$$
$$+\frac{1}{2}\sigma_r^2 r(t)H_{rr} + \alpha[\delta - \eta(t)]H_\eta + \frac{1}{2}\sigma_\eta^2\eta(t)H_{\eta\eta}\} = 0 \quad (10)$$

$$H(T,r,\eta,x) = U(x) \quad (11)$$

where $H_t, H_\eta, H_x, H_r, H_{xx}, H_{rr}, H_{\eta\eta}, H_{x\eta}$ and H_{xr} denote first and second orders partial derivatives w.r.t t, r, η and x respectively.

Differentiation of (10) w.r.t. $\pi_s(t)$ and $\pi_l(t)$, we obtain

$$(\upsilon\eta + \sigma_s\lambda_r k_1 r)H_x + (\pi_s(t)\eta + (\pi_s(t)\sigma_s^2\sigma_r^2 r + \pi_l(t)b_1\sigma_s\sigma_r^2 r)H_{xx} - \sigma_s\sigma_r^2 rH_{xr} + \rho\sigma_\eta\eta H_{x\eta} = 0 \quad (12)$$

and

$$b_1\lambda_r k_1 rH_x + (\pi_s(t)b_1\sigma_s\sigma_r^2 r + \pi_l(t)b_1^2\sigma_r^2 r)H_{xx} - b_1\sigma_r^2 rH_{xr} = 0 \quad (13)$$

Solving (12) and (13) for $\pi_s(t)$ and $\pi_l(t)$ we obtained the optimal investment policy $\left(\pi_s^*(t), \pi_s^*(t)\right)$. From equation (13),

$$\pi_l(t) = \frac{H_{xr}}{b_1 H_{xx}} - \frac{\lambda_r k_1 H_x}{b_1 \sigma_r^2 H_{xx}} - \frac{\pi_s(t)\sigma_s}{b_1} \quad (14)$$

Substituting for $\pi_l(t)$ in equation (12) and simplifying, we obtain

$$\pi_s^*(t) = -\upsilon \frac{H_x}{H_{xx}} - \rho\sigma_\eta \frac{H_{x\eta}}{H_{xx}} \quad (15)$$

Substituting (16) in (14) gives

$$\pi_l^*(t) = \frac{H_{xr}}{b_1 H_{xx}} + \frac{(\upsilon\sigma_s\sigma_r^2 - \lambda_r k_1)H_x}{b_1 \sigma_r^2 H_{xx}} + \frac{\rho\sigma_\eta\sigma_s H_{x\eta}}{b_1 H_{xx}} \quad (16)$$

Substituting (15) and (16) in (10) gives the value function partial differential equation (PDE)

$$H_t + xrH_x - \left(\frac{\upsilon^2\eta}{2} + \frac{\lambda_r^2 k_1^2 r}{2\sigma_r^2}\right)\frac{H_x^2}{H_{xx}} - \rho^2\sigma_\eta^2\eta\frac{H_{x\eta}^2}{2H_{xx}} - \sigma_r^2 r\frac{H_{xr}^2}{2H_{xx}} - \rho\sigma_\eta\eta\upsilon\frac{H_x H_{x\eta}}{H_{xx}} + \lambda_r k_1 r\frac{H_x H_{xr}}{H_{xx}} + (a - br)H_r + \frac{1}{2}\sigma_r^2 r H_{rr} + \alpha(\delta - \eta)H_\eta + \frac{1}{2}\sigma_\eta^2\eta H_{\eta\eta} = 0 \quad (17)$$

Next we solve (18) for $H(t, r, \eta, x)$.

3.0 THE PORTFOLIO OPTIMIZATION PROBLEM SOLUTION

For the case of CRRA utility function, let the solution of equation (18) takes the form:

$$H(t, r, \eta, x) = \frac{x^\beta}{\beta} f(t, r, \eta), \quad \beta < 1, \beta \neq 0 \quad (18)$$

With the boundary condition:

$$f(T, r, \eta) = 1 \quad (19)$$

From (19), we have

$$\left.\begin{aligned} H_t &= \frac{x^\beta}{\beta} f_t, H_x = x^{\beta-1} f, \quad H_r = \frac{x^\beta}{\beta} f_r, \\ H_\eta &= \frac{x^\beta}{\beta} f_\eta, H_{xx} = (\beta - 1)x^{\beta-2} f \\ H_{xr} &= x^{\beta-1} f_r, H_{x\eta} = x^{\beta-1} f_\eta, \\ H_{rr} &= \frac{x^\beta}{\beta} f_{rr}, \quad H_\eta = \frac{x^\beta}{\beta} f_{\eta\eta} \end{aligned}\right\} \quad (20)$$

Introducing these derivatives in (20) into (17) gives and dividing through by $\frac{x^\beta}{\beta}$ yields

$$f_t + \left[r\beta - \left(\frac{\beta\upsilon^2\eta}{2(\beta - 1)} + \frac{\beta\lambda_r^2 k_1^2 r}{2\sigma_r^2(\beta - 1)}\right)\right] f - \frac{\beta\rho^2\sigma_\eta^2\eta f_\eta^2}{2(\beta - 1)f} - \frac{\beta\sigma_r^2 r f_r^2}{2(\beta - 1)f} + \left[\alpha(\delta - \eta) - \frac{\beta\rho\sigma_\eta\eta\upsilon}{\beta - 1}\right] f_\eta + \left[\frac{\beta\lambda_r k_1 r}{\beta - 1} + (a - br)\right] f_r + \frac{1}{2}\sigma_r^2 r f_{rr} + \frac{1}{2}\sigma_\eta^2\eta f_{\eta\eta} = 0 \quad (21)$$

We conjecture $f(t, r, \eta)$ as the following:

$$\left.\begin{aligned} f(t, r, \eta) &= e^{P_1(t) + P_2(t)r + P_3(t)\eta} \\ P_1(t) &= P_2(t) = P_3(t) = 0 \end{aligned}\right\} \quad (22)$$

From (22), we have

$$\left.\begin{aligned} f_t &= (P_1'(t) + P_2'(t)r + P_3'(t)\eta)f \\ f_r &= P_2(t)f, \quad f_\eta = P_3(t)f \\ f_{rr} &= P_2^2(t)f, \quad f_{\eta\eta} = P_3^2(t)f \end{aligned}\right\} \quad (23)$$

Hence substituting for f_t, f_r, f_η, f_{rr} and $f_{\eta\eta}$ in (21) gives

$$[P_1'(t) + aP_2(t) + \alpha\delta P_3(t)]f + rf\left[P_2'(t) + \left(\beta - \frac{\beta\lambda_r^2 k_1^2}{2\sigma_r^2(\beta - 1)}\right) + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{2(\beta - 1)}\right)P_2^2(t) + \left(\frac{\beta\lambda_r k_1}{\beta - 1} - b\right)P_2(t)\right] + \eta f\left[P_3'(t) - \left(\frac{\beta\upsilon^2}{2(\beta - 1)}\right) + \left(\frac{1}{2}\sigma_\eta^2 - \frac{\beta\rho^2\sigma_\eta^2}{2(\beta - 1)}\right)P_3^2(t) - \left(\alpha + \frac{\beta\rho\sigma_\eta\upsilon}{\beta - 1}\right)P_3(t)\right] = 0 \quad (24)$$

Eliminating the dependency on r and η, we decompose (24) into

$$P_1'(t) + aP_2(t) + \alpha\delta P_3(t) = 0 \quad (25)$$

$$P_2'(t) + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{2(\beta - 1)}\right)P_2^2(t) + \left(\frac{\beta\lambda_r k_1}{\beta - 1} - b\right)P_2(t) + \left(\beta - \frac{\beta\lambda_r^2 k_1^2}{2\sigma_r^2(\beta - 1)}\right) = 0 \quad (26)$$

$$P_3'(t) + \left(\frac{1}{2}\sigma_\eta^2 - \frac{\beta\rho^2\sigma_\eta^2}{2(\beta - 1)}\right)P_3^2(t)$$

$$-\left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right)P_3(t)$$
$$-\left(\frac{\beta\upsilon^2}{2(\beta-1)}\right)=0 \qquad (27)$$

Observe that (26) and (27) are the general Riccati equations. From (25),

$$P_1'(t)=-aP_2(t)-\alpha\delta P_3(t)$$
$$P_1(t)=-\left(a\int_t^T P_2(t)\,dt + \alpha\delta\int_t^T P_3(t)\,dt\right) \qquad (28)$$

From (26),

$$\frac{dP_2(t)}{dt}=\left(\frac{\beta\sigma_r^2}{2(\beta-1)}-\frac{1}{2}\sigma_r^2\right)P_2^2(t) + \left(b-\frac{\beta\lambda_r k_1}{\beta-1}\right)P_2(t) + \left(\frac{\beta\lambda_r^2k_1^2}{2\sigma_r^2(\beta-1)}-\beta\right) \qquad (29)$$

The discriminant $= B^2-4AC$

$$=b^2+\frac{\beta(2b\lambda_r k_1-2\sigma_r^2-\lambda_r^2k_1^2)}{1-\beta}$$

since $\beta<1$

where $A=\left(\frac{\beta\sigma_r^2}{2(\beta-1)}-\frac{1}{2}\sigma_r^2\right)$,

$B=\left(b-\frac{\beta\lambda_r k_1}{\beta-1}\right), C=\left(\frac{\beta\lambda_r^2k_1^2}{2\sigma_r^2(\beta-1)}-\beta\right)$

let $Q_1=b^2+\frac{\beta(2b\lambda_r k_1-2\sigma_r^2-\lambda_r^2k_1^2)}{1-\beta}$, then

$$L_{1,2}=\frac{\left(b+\frac{\beta\lambda_r k_1}{1-\beta}\right)\pm\sqrt{Q_1}}{\left(\frac{\sigma_r^2}{1-\beta}\right)}, \qquad \beta<1$$

Equation (27) has different solutions depending on whether $Q_1>0, Q_1=0$ and $Q_1<0$. Now, let $Q_1>0$ then equation (29) have two different roots denoted by L_1 and L_1 respectively such that

$$\frac{dP_2(t)}{dt}=A[(P_2(t)-L_1)(P_2(t)-L_2)] \qquad (30)$$

Therefore, equation (30) becomes

$$\frac{dP_2(t)}{(P_2(t)-L_1)(P_2(t)-L_2)}=Adt \qquad (31)$$

Observe that from (31),

$$\frac{1}{L_1-L_2}\left(\frac{1}{P_2(t)-L_1}-\frac{1}{P_2(t)-L_2}\right)dP_2(t) = Adt \qquad (32)$$

The integral of (32) with respect to t, from t to T is:

$$\frac{1}{L_1-L_2}\int_t^T\left(\frac{dP_2(s)}{P_2(s)-L_1}-\frac{dP_2(s)}{P_2(s)-L_2}\right) = A\int_t^T ds$$

$$P_2(t)=\frac{L_1L_2-L_1L_2e^{A(L_1-L_2)(T-t)}}{M_1-M_2e^{A(L_1-L_2)(T-t)}}$$

Note that

$$A=\left(\frac{\beta\sigma_r^2}{2(\beta-1)}-\frac{1}{2}\sigma_r^2\right)=-\left(\frac{\beta\sigma_r^2}{2(1-\beta)}+\frac{1}{2}\sigma_r^2\right)$$

for $\beta<1$

Therefore,

$$P_2(t) = \frac{L_1L_2-L_1L_2e^{-\left(\frac{1}{2}\sigma_r^2+\frac{\beta\sigma_r^2}{2(1-\beta)}\right)(L_1-L_2)(T-t)}}{L_1-L_2e^{-\left(\frac{1}{2}\sigma_r^2+\frac{\beta\sigma_r^2}{2(1-\beta)}\right)(L_1-L_2)(T-t)}} \qquad (33)$$

Next we solve for $P_3(t)$ in (27)

$$P_3'(t)=\left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}-\frac{1}{2}\sigma_\eta^2\right)P_3^2(t) + \left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right)P_3(t) + \left(\frac{\beta\upsilon^2}{2(\beta-1)}\right) \qquad (34)$$

From (34), we have

$A_1=\left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)}-\frac{1}{2}\sigma_\eta^2\right)$,

$B_1=\left(\alpha+\frac{\beta\rho\sigma_\eta\upsilon}{\beta-1}\right), C_1=\left(\frac{\beta\upsilon^2}{2(\beta-1)}\right)$

The discriminant $=B_1^2-4A_1C_1$

$$=\alpha^2-\frac{2\beta\rho\sigma_\eta\upsilon\alpha}{1-\beta}-\frac{\beta\upsilon^2\sigma_\eta^2}{1-\beta},$$

$\beta<1$

Again, let $Q_2=\alpha^2-\frac{2\beta\rho\sigma_\eta\upsilon\alpha}{1-\beta}-\frac{\beta\upsilon^2\sigma_\eta^2}{1-\beta}$

Then,

$$L_{3,4}=\frac{\left(\alpha-\frac{\beta\rho\sigma_\eta\upsilon}{1-\beta}\right)\pm\sqrt{Q_2}}{\left(\sigma_\eta^2+\frac{\beta\rho^2\sigma_\eta^2}{1-\beta}\right)}, \qquad \beta<1$$

Equation (27) has different solution depending on whether $Q_2>0, Q_2=0$ and $Q_2<0$. Let $Q_2>0$, then equation (34) have two different roots denoted by L_3 and L_4 respectively such that

$$\frac{dP_3(t)}{dt}=A_1[(P_3(t)-L_3)(P_3(t)-L_4)] \qquad (35)$$

$$\frac{dP_3(t)}{(P_3(t)-L_3)(P_3(t)-L_4)} = A_1 dt \qquad (36)$$

$$\frac{1}{L_3-L_4}\left(\frac{1}{P_3(t)-L_3}-\frac{1}{P_3(t)-L_4}\right)dP_3(t) = A_1 dt \qquad (37)$$

The integral of (37) from t to T with respect to t is:

$$\frac{1}{L_3-L_4}\int_t^T\left(\frac{1}{P_3(s)-L_3}-\frac{1}{P_3(s)-L_4}\right)dP_3(s) = A_1\int_t^T ds$$

$$P_3(t) = \frac{L_3L_4 - L_3L_4e^{A_1(L_3-L_4)(T-t)}}{M_3 - M_4e^{A_1(L_3-L_4)(T-t)}}$$

Observe that

$$A_1 = \left(\frac{\beta\rho^2\sigma_\eta^2}{2(\beta-1)} - \frac{1}{2}\sigma_\eta^2\right) = -\left(\frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)} + \frac{1}{2}\sigma_\eta^2\right), \ \ \beta < 1$$

Therefore,

$$P_3(t) = \frac{L_3L_4 - L_3L_4e^{-\left(\frac{1}{2}\sigma_\eta^2+\frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)}\right)(L_3-L_4)(T-t)}}{L_3 - L_4e^{-\left(\frac{1}{2}\sigma_\eta^2+\frac{\beta\rho^2\sigma_\eta^2}{2(1-\beta)}\right)(L_3-L_4)(T-t)}} \qquad (38)$$

Theorem 4.1

From Eq. (15), (16), (20) and (23), the optimal proportion of capital spent in the three assets which are; security, loan and treasury under stochastic interest rates and stochastic instability for the situation of CRRA utility function is specified by:

$$\pi_{sp}^*(t) = \frac{\nu}{1-\beta} + \frac{\rho\sigma_\eta P_3(t)}{1-\beta}$$

$$\pi_{lp}^*(t) = \frac{(\lambda_r k_1 - \upsilon\sigma_s\sigma_r^2)}{b_1\sigma_r^2(1-\beta)} - \frac{P_2(t)}{b_1(1-\beta)} - \frac{\rho\sigma_s\sigma_\eta P_3(t)}{b_1(1-\beta)}$$

$$\pi_{0p}^*(t) = 1 - \pi_{sp}^*(t) - \pi_{lp}^*(t) = 1 + \frac{\upsilon\sigma_s\sigma_r^2 - \upsilon b_1\sigma_r^2 - \lambda_r k_1}{b_1\sigma_r^2(1-\beta)} + \frac{1}{b_1(1-\beta)}P_2(t) + \frac{\rho\sigma_\eta(\sigma_s - b_1)}{b_1(1-\beta)}P_3(t)$$

4.0 NUMERICAL EXAMPLES

Here, we give some numerical examples that present the illustration of the optimum investment policy dynamics with time, risk aversion and market price of risk parameters effect on the optimum investment policy. We assume that the investment period $T = 10$ years. The remaining parameters are $: a = 0.0187, b = 0.2339, r_0 = 0.05, \eta_0 = 1, \beta = -2, \lambda_r = 1, k_1 = 0.0073, \sigma_r = 0.0854, \alpha = 2, \delta = 0.3, \rho = 0.5, \sigma_\eta = 1, \upsilon = 1.5, b_1 = 0.7, \sigma_s = 0.02$

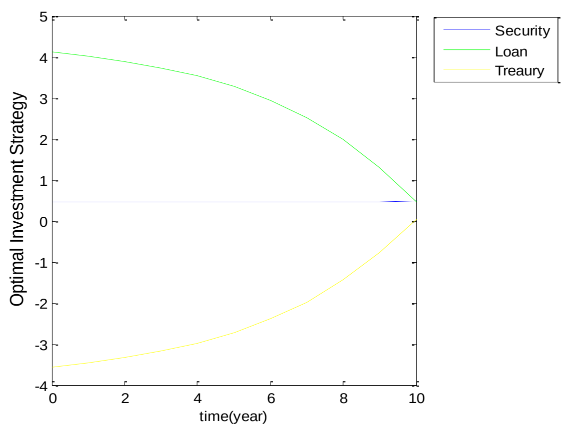

Fig. 1: Time effect on the optimum investment policy

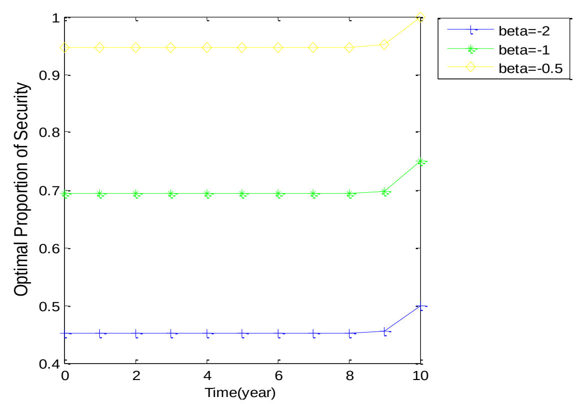

Fig. 2: Parameter β effect on the Security

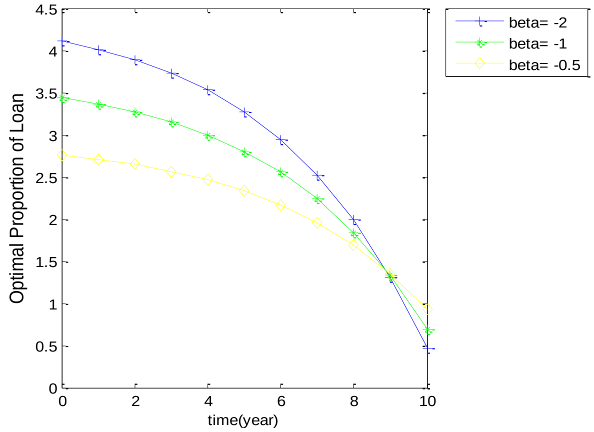

Fig. 3: Parameter β effect on Loan

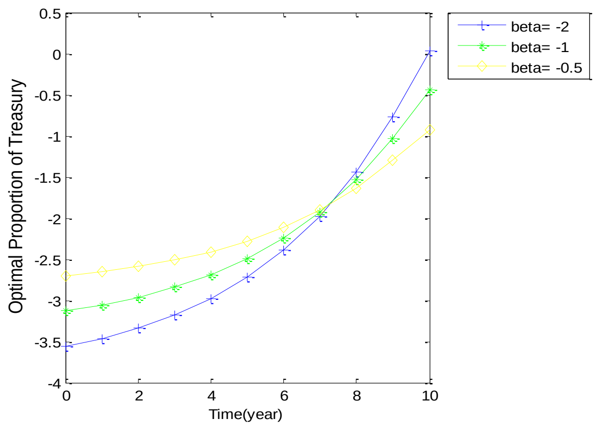

Fig. 4: Parameter β effect on Treasury

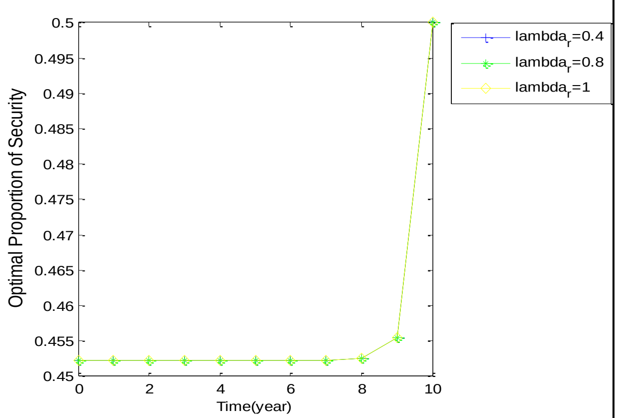

Fig. 5: Parameter λ_r effect on security

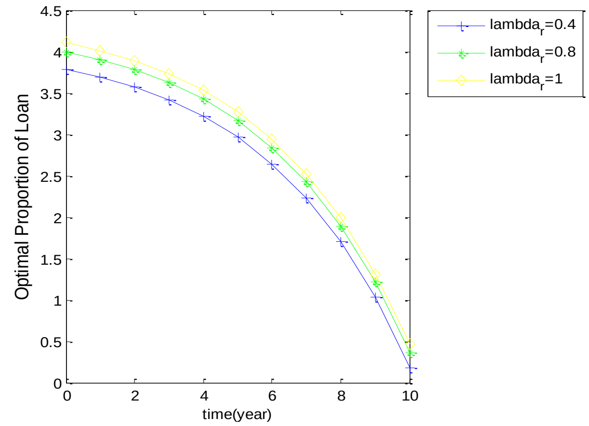

Fig. 6: Parameter λ_r effect on Loan

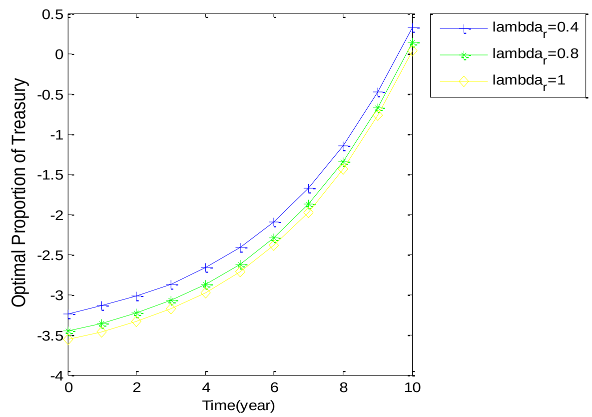

Fig. 7: Parameter λ_r effect on Treasury

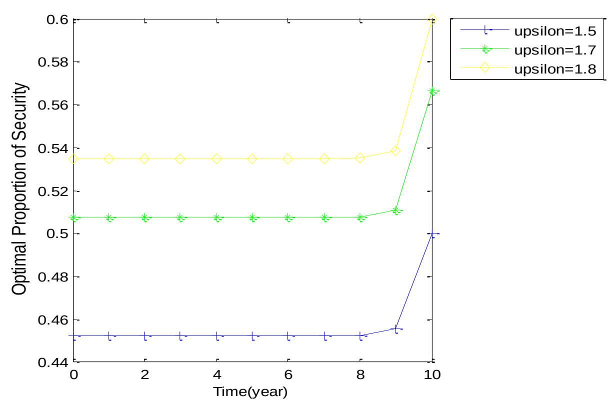

Fig. 8: Parameter υ effect on Security

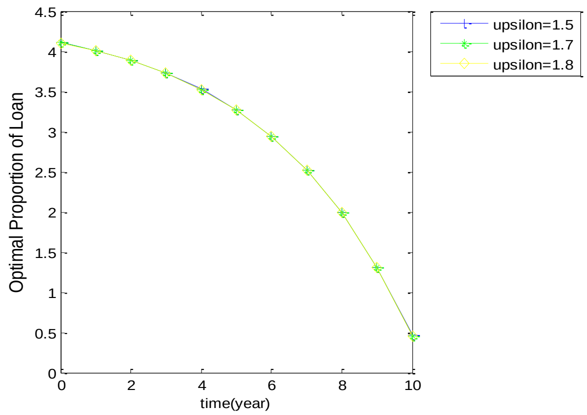

Fig. 9: Parameter υ effect on Loan

Figure 1 illustrates the trends of how the optimal proportion of the total capital put in the three assets changes with time. From figure 1, there is a positive relationship between optimal investment in the treasury and time. That is, as time increases so also the optimal investment in the treasury. However, the optimal quantity invested in the security almost remains unchanged and the optimal quantity devoted to loan decreases. Figure 1 also shows that the optimal quantity put in treasury is negative at the beginning of the investment time which indicates the investor short position in treasury. The treasury short position within this investment period enables the investor to invest in the risky instruments but as the investment period approaches the end, the investor invests more in the treasury to reach the optimum investment policy.
The utility function is:

$$u(x) = \frac{x^\beta}{\beta}$$

Therefore, $1 - \beta$ is the risk aversion coefficient. It indicates that the bigger the value of β, the lesser the relative risk aversion. Therefore, the more the investor would like to invest in (security and loan) referred to as the risky assets. Also, optimal quantity invested in security and loan increases as the parameter β increases as shown in Figure 2 and 3 while optimum quantity invested in treasury decreases as shown in Figure 4.

Figure 5 to Figure 7 illustrate the correlation between the optimum investment strategy or policy and the risk parameter for market price, λ_r for loan. The investor invests more in the treasury at first but the proportion invested in the treasury decreases as

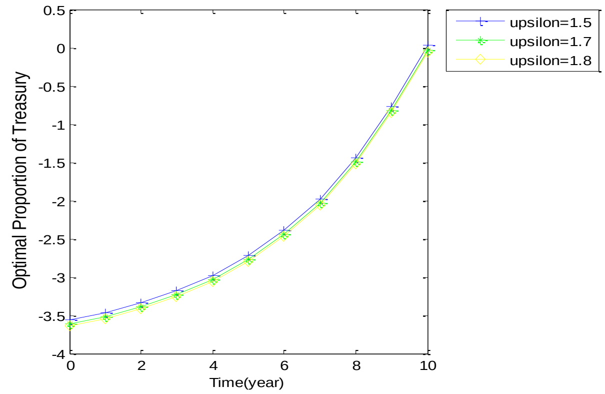

Fig. 10: Parameter υ effect on Treasury

λ_r increases and this is shown in figure 7. The optimal proportion invested in the security remains unchanged as shown in figure 5 but the optimal quantity put in loan increases as λ_r increases as illustrated in figure 6 which is intuitive from (3). The expression $b_1\lambda_r k_1 r(t)$ is the appreciation rate of the loan, therefore investment in loan increases as λ_r increases.

The proportion of optimal amount invested in security increases as market price of risk parameter, υ for security increases. Observe from (4) that $\upsilon\eta(t) + \sigma_s\lambda_r k_1 r(t)$ is the rate of increase of the security. Therefore, as the parameter υ rises, investment in security rises. A numerical example is illustrated in Figure 8. The proportion of amount invested in the loan remains unchanged as in Figure 9 while the proportion of optimal amount invested in treasury decreases as the result of increase of investment in security as illustrated in figure 10.

5.0 CONCLUSION

Here, we have looked at how a financial institution can optimally allocates its wealth among its assets namely; security, loan and treasury in the setting where interest rate and volatility are stochastic. We also derived the optimum investment policy for the financial institution for the case of CRRA utility function. We also studied the time effect, risk aversion parameter effect and market price of risk parameter effect on the optimum investment policy through numerical examples. From the results obtained from the numerical examples, we have:

i. The optimum investment policy can be achieved by decreasing the investment of the financial institution in (security and loan) and increasing it in treasury as the investment period approaches the end.

ii. The investor invests more in security and loan as the investor adopts less risk averse policy.

iii. As the reward of the risks associated with security and loan increase, the more the investor invests in the risky assets.

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