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Single-Step Sixth Stage Implicit Runge-Kutta Method for First-Order
Initial Value Problems of Ordinary Differential Equations

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ABSTRACT

We present a single-step block hybrid method obtained through multistep collocation (MC) approach, by incorporating three off-grid interpolation and two off-grid collocation points in a continuous linear hybrid method as opposed to a similar work where two-step methods were obtained. The discrete hybrid methods are used in the formulation a block method which is then reformulated into a new single-step sixth-stage implicit Runge-Kutta method (SIRK) for the numerical treatment of first-order ordinary differential equations. The basic convergence and A-stability properties of the new method are established. Numerical experiments performed using the new method further revealed a reduction in the absolute errors, showing that this method is a better candidate for similar existing ones in the literature, as such it should be used for such class of problems.

Keywords: block hybrid method, Butcher tableau, implicit Runge-Kutta, multistep collocation, ODEs.

1.0 INTRODUCTION

It is common knowledge that most studies in engineering, science, economics and many areas of the physical world often results in ordinary differential equations (ODEs). Since not all model ODEs have theoretical solutions, the ability to obtain accurate numerical solutions is important and it shall continue to dominate the attention of researchers every now and then.

The general first-order initial value problem (IVP) for ODEs takes the form

$$y'(x) = f(x, y), y(a) = y_0 \quad (1)$$

on the interval $[a \leq x \leq b]$ where the function f is continuous and the existence and uniqueness of the solution for (1) is assumed.

1.2 Statement of the Problem

Adee *et al.* (2022) studied (1) where a pair of two-step implicit Runge-Kutta (RK) type methods were derived and implemented with better numerical accuracy as opposed to similar two-step or higher step methods in the literature.

To this effect, this research further extends the strength of the results in Adee *et al.* (2002) to further reduce the step number from two to one, thereby reducing the computational burden of implementation and at the same, improving the accuracy of the results.

The approach uses a block hybrid method which is basically a set of linear multistep methods which are modified to incorporate off-grid points. Furthermore, in this research, like Agam and Yahaya (2014), Yahaya and Adegboye (2013), Zamurat and Ummikusum (2014), Muhammad, Yahaya and Abdulkareem (2015) and Mshelia, *et al.* (2016), the resulting method is not implemented as a set of linear multistep methods but reformulated into a type of implicit Runge-Kutta method which simultaneous yields approximate solution to (1).

Moreover, single-step methods such as RK methods are by default self starting as they do not require extra methods to commence the solution. Furthermore, literature abounds that implicit

numerical methods incur more cost in their derivation and implementation, however enjoy better accuracy and stability properties than their explicit counterparts, hence even in this research, these qualities are employed.

2.0. MATERIALS AND METHOD

2.1 Development of a New Single-Step Method.

We use multistep collocation technique to develop a new single-step block hybrid method for the numerical solution of (1).

Following Adey *et al.* (2022) and Mshelia, *et al.* (2016), a continuous solution of (1) with r interpolation points (grid and/off-grid) and m collocation points (grid and/off-grid) points takes the form

$$y(x) = \sum_{i=0}^{p-1} \left(\sum_{j=0}^{r-1} \alpha_{j,i+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,i+1} f_{n+j} \right) x^i$$

$$= \sum_{i=0}^{p-1} \xi_i x^i \tag{2}$$

where

$$\xi_i = \sum_{j=0}^{r-1} \alpha_{j,i+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,i+1} f_{n+j} \tag{3}$$

From (2),

$$y'(x) = \sum_{i=0}^{p-1} i \xi_i x^{i-1} \tag{4}$$

In this article, (2) has $r=4$ $(x_{n+r}, r=0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5})$ indicating three off-grid points at $r = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $m=3$ collocation points $(x_{n+m}, m = \frac{3}{5}, \frac{4}{5}, 1)$ which is, two off-grid points at $r = \frac{3}{5}, \frac{4}{5}$ in (4).

The resulting matrix equation is of the form

$$DA = Y \tag{5}$$

where D is of dimension $(r+m) \times (r+m)$ and is given by

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{5}} & x_{n+\frac{1}{5}}^2 & x_{n+\frac{1}{5}}^3 & x_{n+\frac{1}{5}}^4 & x_{n+\frac{1}{5}}^5 & x_{n+\frac{1}{5}}^6 \\ 1 & x_{n+\frac{2}{5}} & x_{n+\frac{2}{5}}^2 & x_{n+\frac{2}{5}}^3 & x_{n+\frac{2}{5}}^4 & x_{n+\frac{2}{5}}^5 & x_{n+\frac{2}{5}}^6 \\ 1 & x_{n+\frac{3}{5}} & x_{n+\frac{3}{5}}^2 & x_{n+\frac{3}{5}}^3 & x_{n+\frac{3}{5}}^4 & x_{n+\frac{3}{5}}^5 & x_{n+\frac{3}{5}}^6 \\ 0 & 1 & 2x_{n+\frac{3}{5}} & 3x_{n+\frac{3}{5}}^2 & 4x_{n+\frac{3}{5}}^3 & 5x_{n+\frac{3}{5}}^4 & 6x_{n+\frac{3}{5}}^5 \\ 0 & 1 & 2x_{n+\frac{4}{5}} & 3x_{n+\frac{4}{5}}^2 & 4x_{n+\frac{4}{5}}^3 & 5x_{n+\frac{4}{5}}^4 & 6x_{n+\frac{4}{5}}^5 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 \end{pmatrix}$$

and

$$A = (a_0, a_0, a_2, a_3, a_4, a_5, a_6)^T$$

$$Y = (y_n, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1})^T$$

We then solve for A in (5) using Maple 18 software and substitute the result into (2) with

$t = \frac{x-x}{h}$ to obtain a one-step continuous hybrid method of the form

$$y(t) = \sum_{j=0}^3 \alpha_j(t) y_{n+\frac{j}{5}} + h \left(\sum_{j=3}^5 \beta_j(t) f_{n+\frac{j}{5}} \right) \tag{6}$$

where

$$\alpha_0(t) = \frac{1290625}{35244} t^6 - \frac{403750}{2937} t^5 + \frac{3630125}{17622} t^4 - \frac{41900}{267} t^3 + \frac{2243545}{35244} t^2 - \frac{37870}{2937} t + 1$$

$$\alpha_{\frac{1}{5}}(t) = -\frac{509375}{1958} t^6 + \frac{3624375}{3916} t^5 - \frac{2508375}{1958} t^4 + \frac{306725}{356} t^3 - \frac{550995}{1958} t^2 + \frac{34920}{979} t$$

$$\alpha_{\frac{2}{5}}(t) = \frac{3624375}{3916} t^6 - \frac{3078750}{979} t^5 + \frac{7889625}{1958} t^4 - \frac{215900}{89} t^3 + \frac{2626815}{3916} t^2 - \frac{63270}{979} t$$

$$\alpha_{\frac{3}{5}}(t) = -\frac{12528125}{17622} t^6 + \frac{27686875}{11748} t^5 - \frac{52061375}{17622} t^4 + \frac{1838225}{1068} t^3 - \frac{7983485}{17622} t^2 + \frac{122920}{2937} t$$

$$\beta_{\frac{3}{5}}(t) = -\frac{66875}{1958} t^6 + \frac{97375}{979} t^5 - \frac{105725}{979} t^4 + \frac{4870}{89} t^3 - \frac{25131}{1958} t^2 + \frac{1071}{979} t$$

$$\beta_1(t) = \frac{23125}{3916} t^6 - \frac{61625}{3916} t^5 + \frac{62825}{3916} t^4 - \frac{2765}{356} t^3 + \frac{3453}{1958} t^2 - \frac{144}{979} t$$

Evaluating (6) at the points $t = \frac{4}{5}, 1$ and its first

derivative at the points $t = 0, \frac{1}{5}, \frac{2}{5}$ respectively yields the following five discrete hybrid schemes

$$y_{n+\frac{4}{5}} - \frac{2792}{4895}y_{n+\frac{3}{5}} - \frac{2484}{4895}y_{n+\frac{2}{5}} + \frac{424}{4895}y_{n+\frac{1}{5}} - \frac{43}{4895}y_n = h \left(\frac{4728}{24475}f_{n+\frac{4}{5}} + \frac{2004}{24475}f_{n+\frac{3}{5}} - \frac{72}{24475}f_{n+1} \right)$$

$$y_{n+1} - \frac{1420}{979}y_{n+\frac{3}{5}} + \frac{560}{979}y_{n+\frac{2}{5}} + \frac{-135}{979}y_{n+\frac{1}{5}} + \frac{16}{979}y_n = h \left(\frac{-24}{979}f_{n+\frac{3}{5}} + \frac{288}{979}f_{n+\frac{2}{5}} + \frac{60}{979}f_{n+1} \right)$$

$$-122920y_{n+\frac{3}{5}} + 189810y_{n+\frac{2}{5}} - 104760y_{n+\frac{1}{5}} + 3787y_n = h \left(-2937f_n - 18624f_{n+\frac{3}{5}} + 3213f_{n+\frac{2}{5}} - 432f_{n+1} \right)$$

(7)

$$-144970y_{n+\frac{3}{5}} + 266760y_{n+\frac{2}{5}} - 111510y_{n+\frac{1}{5}} - 10280y_n = h \left(14685f_{n+\frac{3}{5}} - 19824f_{n+\frac{2}{5}} + 2928f_{n+\frac{1}{5}} - 369f_{n+1} \right)$$

$$-106595y_{n+\frac{3}{5}} + 87885y_{n+\frac{2}{5}} + 20115y_{n+\frac{1}{5}} - 1405y_n = h \left(-14685f_{n+\frac{3}{5}} - 11109f_{n+\frac{2}{5}} + 1143f_{n+\frac{1}{5}} - 129f_{n+1} \right)$$

Writing (7) in the normalized block form gives

$$A_0 Y_s = A_1 Y_{s-1} + h(B_0 F_{s-1} + B_1 F_s) \quad (8)$$

where s represents the block number,

$$Y_s = (y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1})^T,$$

$$Y_{s-1} = (y_{n-\frac{4}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{1}{5}}, y_n)^T,$$

$$F_s = (f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1})^T \text{ and}$$

$$F_{s-1} = (f_{n-\frac{4}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{1}{5}}, f_n)^T,$$

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{19}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{51}{800} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{19}{228} \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} \frac{1427}{7200} & -\frac{133}{1200} & \frac{241}{3600} & -\frac{173}{7200} & \frac{3}{800} \\ \frac{43}{150} & \frac{7}{225} & \frac{7}{225} & -\frac{1}{75} & \frac{1}{450} \\ \frac{219}{800} & \frac{57}{400} & \frac{57}{400} & -\frac{21}{800} & \frac{3}{800} \\ \frac{64}{225} & \frac{8}{75} & \frac{64}{225} & \frac{14}{225} & 0 \\ \frac{25}{96} & \frac{25}{144} & \frac{25}{144} & \frac{25}{96} & \frac{19}{288} \end{pmatrix}$$

are 5×5 matrices.

We represent (8) using A_0, A_1, B_0 and B_1 in the

Butcher tableau of the form

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \quad (9)$$

where $A = a_{i,j}, b = b_i$

and

$$c = c_i (i = 1, 2, \dots, s)$$

$$= \sum_{j=1}^s a_{i,j} \text{ yields}$$

0	:	0	0	0	0	0	0
$\frac{1}{5}$:	$\frac{19}{288}$	$\frac{1427}{7200}$	$-\frac{133}{1200}$	$\frac{241}{3600}$	$-\frac{173}{7200}$	$\frac{3}{800}$
$\frac{2}{5}$:	$\frac{14}{225}$	$\frac{43}{150}$	$\frac{7}{225}$	$\frac{7}{225}$	$-\frac{1}{75}$	$\frac{1}{450}$
$\frac{3}{5}$:	$\frac{51}{800}$	$\frac{219}{800}$	$\frac{57}{400}$	$\frac{57}{400}$	$-\frac{21}{800}$	$\frac{3}{800}$
$\frac{4}{5}$:	$\frac{14}{225}$	$\frac{64}{225}$	$\frac{8}{75}$	$\frac{64}{225}$	$\frac{14}{225}$	0
1	:	$\frac{19}{228}$	$\frac{25}{96}$	$\frac{25}{144}$	$\frac{25}{144}$	$\frac{25}{96}$	$\frac{19}{288}$
...
1	:	$\frac{19}{228}$	$\frac{25}{96}$	$\frac{25}{144}$	$\frac{25}{144}$	$\frac{25}{96}$	$\frac{19}{288}$

Expressing (10) as an implicit Runge-Kutta type method yields

$$y_{n+1} = y_n + h \begin{pmatrix} \frac{19}{288}k_1 + \frac{25}{96}k_2 + \frac{25}{144}k_3 + \frac{25}{144}k_4 \\ \frac{25}{96}k_5 + \frac{19}{288}k_6 \end{pmatrix} \quad (11)$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{1}{5}h, y_n + h\left(\frac{19}{288}k_1 + \frac{1427}{7200}k_2 - \frac{133}{1200}k_3 + \frac{241}{3600}k_4 - \frac{173}{7200}k_5 + \frac{3}{800}k_6\right)\right)$$

$$k_3 = f\left(x_n + \frac{2}{5}h, y_n + h\left(\frac{14}{225}k_1 + \frac{43}{150}k_2 + \frac{7}{225}k_3 + \frac{7}{225}k_4 - \frac{1}{75}k_5 + \frac{1}{450}k_6\right)\right)$$

$$k_4 = f\left(x_n + \frac{3}{5}h, y_n + h\left(\frac{51}{800}k_1 + \frac{219}{800}k_2 + \frac{57}{400}k_3 + \frac{57}{400}k_4 - \frac{21}{800}k_5 + \frac{3}{800}k_6\right)\right)$$

$$k_5 = f\left(x_n + \frac{4}{5}h, y_n + h\left(\frac{14}{225}k_1 + \frac{64}{225}k_2 + \frac{8}{75}k_3 + \frac{64}{225}k_4 + \frac{14}{225}k_5 + 0k_6\right)\right)$$

$$k_6 = f\left(x_n + h, y_n + h\left(\frac{19}{288}k_1 + \frac{25}{96}k_2 + \frac{25}{144}k_3 + \frac{25}{144}k_4 + \frac{25}{96}k_5 + \frac{19}{288}k_6\right)\right)$$

The method (11) is the new single-step sixth stage implicit Runge-Kutta (SIRK) method.

3.0 RESULTS AND DISCUSSION

3.1 Convergence analysis of SIRK (11) method.

Theorem 1: Super-convergence (Adee et al. 2022).

If the condition $B(p)$ holds for some $p \geq s$, then collocation method has order P where

$$B(p) = \sum_{i=1}^s b_i c_i^{k-1} = \frac{1}{k}, k = 1, 2, \dots, p \quad (12)$$

Applying (12) to (10), it is easily seen that the SIRK (11) is of order $p = 6$.

Furthermore, a check reveals that the new method is consistent since $\sum_{i=1}^s b_i = 1$ (Butcher, 2018).

This implies the SIRK (11) is consistent. Hence we assert the convergence of the SIRK (11) method (Butcher, 2018).

3.2 Stability of the Runge-Kutta method (Butcher, 2018).

Definition 1: The stability function $R(z)$ for (9) is defined by

$$R(z) = \frac{\det(I - zA + zeb)}{\det(I - zA)} = \frac{N(z)}{D(z)} \quad (13)$$

where $\{z \in C^- : R(z) \leq 1\}$, $e = (1, 1, \dots, 1)^T$.

Theorem 2: A Runge-Kutta method with stability function $R(z) = \frac{N(z)}{D(z)}$ is A-stable if and only if

(a) all poles of $R(z)$ (that is, all zeros of $D(z)$) are in the right half-plane and (b) $E(y) \geq 0$ for all real y where

$$E(y) = D(iy)D(-iy) - N(iy)N(-iy) \quad (\text{see Butcher, 2018 for the proof}).$$

Applying (13) to (10) yields the stability polynomial

$$R(z) = \frac{\frac{1}{18750}z^5 + \frac{137}{112500}z^4 + \frac{3}{200}z^3 + \frac{17}{150}z^2 + \frac{1}{2}z + 1}{-\frac{1}{18750}z^5 + \frac{137}{112500}z^4 - \frac{3}{200}z^3 + \frac{17}{150}z^2 - \frac{1}{2}z + 1}. \quad \text{It}$$

is obvious that $R(z)$ clearly satisfies theorem 2 and in particular, $E(y) = 0$ for all real y . Thus, SRIK (11) is A-stable and can effectively handle stiff initial value problems as well.

3.3 Numerical experiments.

We tested the accuracy of the new SIRK (11) in

terms of its absolute errors with some numerical examples from the literature, and the results are presented in Tables 1–5. We used the following notations:

$a(b) := a \times 10^b$ and for the methods-

SIRK –Method (11) for step number $k = 1$

AGM –Agam and Yahaya (2014)

JASY –James *et al.* (2013)

KWM –Kwami (2011), $k = 3$

MAFT –Momoh *et al.* (2014)

MET –Mshelia *et al.* (2016), $k = 3$

TSIRK1 –Adee *et al.* (2022), $k = 2$

TSIRK2 –Adee *et al.* (2022), $k = 2$

Example 1 (Source: Mshelia *et al.* 2016). Solve

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.$$

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$$

Exact solution: $y(x) = -\frac{1}{3}e^{-3x} + \frac{13}{12}e^{2x}$. In

Table 1, the results of SIRK are compared with MET, TSIRK1 and TSIRK2 and the smaller errors at each tabulated points in Table 1 indicate that SIRK has better accuracy than any of these methods.

Table 1: Absolute Errors for Example 1.

x	MET	TSIRK1	TSIRK2	SIRK (11)
0.1	1.4442E(-11)	6.8729E(-12)	5.0930E(-12)	4.9614E(-12)
0.2	3.5268E(-11)	1.6785E(-11)	1.2438E(-11)	1.2116E(-11)
0.3	6.4600E(-11)	3.0744E(-11)	2.2782E(-11)	2.2192E(-11)
0.4	1.0518E(-10)	5.0059E(-11)	3.7094E(-11)	3.613E(-11)
0.5	1.6057E(-10)	7.6410E(-11)	5.6625E(-11)	5.5161E(-11)
0.6	2.0000E(-09)	1.1190E(-10)	8.2984E(-11)	8.083E(-11)
0.7	3.0000E(-09)	1.5956E(-10)	1.1824E(-10)	1.1518E(-10)
0.8	1.0000E(-09)	2.2271E(-10)	1.6503E(-10)	1.6076E(-10)
0.9	1.0000E(-09)	3.0600E(-10)	2.2675E(-10)	2.2088E(-10)
1.0	6.0000E(-09)	4.1524E(-10)	3.0770E(-10)	2.9974E(-10)

Example 2 (Source: Agam and Yahaya, 2014).

Solve

$$y' = -8y + 8x + 1, y(0) = 2, h = 0.1, 0 \leq x \leq 1$$

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$$

Exact solution: $y(x) = x + 2e^{-8x}$.

Table 2 compares SIRK result with AGY, TSIRK1 and TSIRK2, and the smaller errors in Table 2 again indicate better accuracy of SIRK.

Table 2: Absolute Errors for example 2.

x	AGY	TSIRK1	TSIRK2	SIRK
0.1	1.86 E(-06)	1.1497E(-07)	9.8582E(-08)	5.8854 E(-08)
0.2	1.69 E(-06)	1.0332E(-07)	8.8591E(-08)	5.2890 E(-08)
0.3	1.14 E(-06)	6.9638E(-08)	5.9710E(-08)	3.5647 E(-08)
0.4	6.80 E(-07)	4.1721E(-08)	3.5772E(-08)	2.1357 E(-08)
0.5	3.85 E(-07)	2.3433E(-08)	2.0092E(-08)	1.1995 E(-08)

Example 3 (Source: Mshelia *et al.* 2016). Solve

$$y' = 20x^2 - 20y + 2x, y(0) = \frac{1}{3}, h = 0.05, 0 \leq x \leq 1$$

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$$

Exact solution: $y(x) = x^2 + \frac{1}{3}e^{-20x}$. The results are compared with MET, TSIRK1 and TSIRK2 in Table 3, and the smaller errors in Table 3 also indicate better performance of SIRK (11) over the others.

Table 3: Absolute Errors for example 3.

x	MET	TSIRK1	TSIRK2	SIRK (11)
0.1	1.2554E(-07)	6.0252E(-08)	5.2655E(-08)	3.77707 E(-16)
0.2	3.3979E(-08)	1.6317E(-08)	1.4252E(-08)	6.6613 E(-16)
0.3	6.8979E(-09)	3.3123E(-09)	2.8932E(-09)	9.9920 E(-16)
0.4	1.2447E(-09)	5.9770E(-10)	5.2207E(-10)	1.2212 E(-15)
0.5	2.1056E(-10)	1.0111E(-10)	8.8319E(-11)	1.3323 E(-15)
0.6	3.4196E(-11)	1.6421E(-11)	1.4343E(-11)	1.5543 E(-15)
0.7	5.4000E(-12)	2.5928E(-12)	2.2647E(-12)	1.5543 E(-15)
0.8	8.3600E(-13)	4.0112E(-13)	3.5039E(-13)	1.5543 E(-15)
0.9	1.2800E(-13)	6.1062E(-14)	5.3291E(-14)	1.5543 E(-15)
1.0	1.0000E(-14)	9.1038E(-15)	7.9936E(-15)	1.6098 E(-15)

Example 4 (Source: Msheli *et al.* 2016). Solve

$$y' = -y, y(0) = 1, h = 0.05, 0 \leq x \leq 1$$

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$$

Exact solution: $y(x) = e^{-x}$

Table 4 compares SIRK with results in MET, TSIRK1 and TSIRK2 and the smaller errors in Table 4 again indicate the improved accuracy of the SIRK.

Table 4: Absolute Errors for example 4

x	MET	TSIRK1	TSIRK2	SIRK
0.1	2.0E(-15)	5.5511E(-16)	4.4409E(-16)	3.77707 E(-16)
0.2	4.0E(-15)	1.1102E(-15)	7.7716E(-16)	6.6613 E(-16)
0.3	6.0E(-15)	1.5543E(-15)	1.1102E(-15)	9.9920 E(-16)
0.4	7.0E(-15)	1.8874E(-15)	1.3323E(-15)	1.2212 E(-15)
0.5	7.0E(-15)	2.1094E(-15)	1.5543E(-15)	1.3323 E(-15)
0.6	7.0E(-15)	2.4425E(-15)	1.7764E(-15)	1.5543 E(-15)
0.7	9.0E(-15)	2.4425E(-15)	1.7764E(-15)	1.5543 E(-15)
0.8	9.0E(-15)	2.4425E(-15)	1.7764E(-15)	1.5543 E(-15)
0.9	9.0E(-15)	2.4980E(-15)	1.8319E(-15)	1.5543 E(-15)
1.0	9.0E(-15)	2.5535E(-15)	1.8874E(-15)	1.6098 E(-15)

Example 5 (Source: Momoh *et al.* 2014). Solve

$$y' = xy, y(0) = 1, h = 0.1, 0 \leq x \leq 1$$

$$y' - 2y = e^{-x}, y(0) = \frac{3}{4}, h = 0.1, 0 \leq x \leq 1.0$$

Exact solution: $y(x) = e^{\frac{x^2}{2}}$.

The results of SIRK are compared with MAFT TSIRK1 and TSIRK2 in Table 5. The smaller errors in Table 5 clearly demonstrate the improved accuracy of SIRK over the others.

Table 5: Absolute Errors for example 5.

x	MAFT	TSIRK1	TSIRK	SIRK
0.1	2.6067E(-11)	2.3648E(-13)	1.9096E(-13)	1.5388 E(-13)
0.2	8.4319E(-11)	9.5901E(-13)	7.5540E(-13)	6.341 E(-13)
0.3	1.8509E(-10)	2.2520E(-12)	1.7577E(-12)	1.4995 E(-12)
0.4	3.4796E(-10)	4.2693E(-12)	3.3151E(-12)	2.8562 E(-12)
0.5	6.0512E(-10)	7.2562E(-12)	5.6146E(-12)	4.8734 E(-12)
0.6	1.0070E(-09)	1.1582E(-11)	8.9369E(-12)	7.8062 E(-12)
0.7	1.6310E(-09)	1.7791E(-11)	1.3696E(-11)	1.2032 E(-11)
0.8	2.5956E(-09)	2.6682E(-11)	2.0499E(-11)	1.8105 E(-11)
0.9	4.0816E(-09)	3.9426E(-11)	3.0230E(-11)	2.6840 E(-11)
1.0	6.3647E(-09)	5.7738E(-11)	4.4190E(-11)	3.9436 E(-11)

4.0 CONCLUSION

All the results in Tables 1–5 indicate that the absolute errors of the SIRK (11) produced the smallest errors when compared with the listed method which indicates better accuracy. This then makes the new method, a good alternative for solving similar ODEs, as opposed to the existing methods in the literature. The A-stability property of the method equally suggests its applicability to stiff ODEs which require special care in their numerical solution..

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