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Mathematical Modeling and Optimal Control of Intervention Strategies for Covid-19 Disease

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ABSTRACT

This research work used mathematical modeling in understanding the dynamics of covid-19 disease. We modified the work of Chen *et al.* (2020) by incorporating vaccination and pets (spread agents) compartments, making the model a ten (10) compartmental model, and also augmenting four controls to it namely: vaccination, use of face mask and physical distancing, sanitation, and treatment. We developed from the model, a system of non-linear Ordinary Differential Equations from which the positivity of solution was proven. We established the equilibrium states, determined the reproduction number R_0 which was utilized to predict the disease's transmission dynamics, hence establishing the conditions for local and global stability of the disease free- equilibrium using Routh-Hurwitz criterion and the Castillo-Chavez technique, respectively. The outcome of the investigation of the stability of the disease-free equilibrium state that covid-19 disease transmission can be significantly degraded and eliminated if the secondary infection's rate R_0 is maintained at a value less than unity. We also used Pontryagin's Maximum Principle to establish the optimality system. The optimality system was numerically solved in Matlab to establish the best strategy in controlling the transmission of covid-19 disease in the population. The graphical solutions revealed that the most effective strategy is the combination of vaccination, use of face mask and physical distancing, sanitation, and treatment of infected individuals in the population.

Keywords: Mathematical Modeling, Covid-19, Transmission, Strategies, Optimal Control

1.0 INTRODUCTION

Covid-19 is a viral disease that is caused by a severe acute respiratory syndrome (SARS-COV-2) (WHO, 2021). It is part of a large family of coronaviruses – a zoonotic virus that transmits from animals to humans. The disease first emerged in December of 2019 in Huanan sea food market of Wuhan city in Hubei province in China. The disease is spread through droplets produced when an infected individual coughs, sneezes, exhales, or touch surfaces. Infected individuals show symptoms as difficulty breathing, coughing, fever, loss of taste and smell, fatigue, nausea, severe headache, etc. (WHO, 2021). Due to its human to human transmission, the disease quickly spreads (exponentially) throughout the world affecting millions of people and became a pandemic as declared by the WHO on March 11, 2020 (WHO, 2020). As at 5:52 pm CEST, 27 May 2022, there

were 525,467,084 confirmed cases of covid-19 disease (WHO, 2022). To mitigate the prevalence of the Covid-19 disease among the populace, governments around the world took drastic measures through the use of some Non-Pharmaceutical Interventions (NPIs) such as lockdown, physical distancing, use of face masks, as the first line of action and desperately so the only then – for there is no definite cure to it. These measures took a wreaking havoc on the socio-economic life of the people as movements and gatherings were restricted for considerable amount of time enough to create instability and economic shocks.

With the Covid-19 becoming widespread, lots of efforts have been put forth around the world to comprehend and predict the dynamics of the disease transmission for control purposes, with

researchers developing mathematical models that mimics the behavior of the disease. Transmission from the infection source (likely bats) to humans, and then human to human were modeled by Chen Yusuf, Kumam, and Yusuf (2021) and many more. Also, given the behavior of the disease and measures employed to curb its spread, models were developed by Zeb, Alzharani, and Zaman (2020); Sasmita (2020) etc. for effective planning and deployment of best strategies to curtailing the spread.

Discoveries of vaccines as announced by the WHO and that some pets were found to carry the virus that causes covid-19 (CDC, 2021) motivated us to develop a model that captures these developments. In this paper, we developed a ten (10) compartmental model comprising of three (3) different populations; the human population, the pets (cats) population, and the population of pathogens in the environment. Also, we augmented four controls in the model namely vaccination (u_1), use of face mask and physical distancing (u_2), sanitation (u_3), and treatment (u_4). From the model, we formulated a system of non-linear ODEs which we used to obtain the invariant region, proved the positivity of the system's solution, determined the equilibrium states (disease free equilibrium and endemic equilibrium) of the system, determined the reproduction number which

$$\frac{dV_H}{dt} = \psi S_H - (\psi + \mu_H) V_H$$

$$\frac{dS_H}{dt} = \Lambda_H + \psi V_H - [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\psi + \mu_H) S_H$$

$$\frac{dE_H}{dt} = [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\omega_H + \mu_H) E_H$$

$$\frac{dI_H}{dt} = (1 - \sigma) \omega_H E_H - (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H$$

$$\frac{dA_H}{dt} = \sigma \omega_H E_H - (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H$$

et al. (2020); Anzum and Islam (2020); Tiwari, Deyal, and Bisht (2020); Sarbaz, Salih, and Sulaimany (2020); Daniel (2020); Ahmed, Modu,

was used to establish the transmission dynamics of the disease, proved the stability of the disease free equilibrium, and most importantly obtained the optimal control strategy using numerical simulations.

The remainder of this paper is organized as follows: section 2 presents the model formulation with constant control. Section 3 presents the positivity of the solution of the system, the invariant region, the equilibrium states of the system, and the basic reproduction number. We discuss the local stability and the global stability of the disease free equilibrium of the system in section 4. We presented the optimal control analysis in section 5. Numerical simulations of the system with variable controls and discussion of numerical results are presented in section 6, and in section 7, we present the conclusions.

2.0 MODEL FORMULATION

We proposed a mathematical model of covid-19 disease based on the R-P model by Chen *et al.* (2020) which describes the transmission network of covid-19 disease, incorporating vaccination and pets (cats) compartments from which we formulated the following system of ODEs:

$$\frac{dH_H}{dt} = \alpha_1 I_H + \alpha_2 A_H - (\delta_{H_3} + \gamma + \mu_H) H_H \quad (1)$$

$$\frac{dR_H}{dt} = \gamma H_H - \mu_H R_H$$

$$\frac{dV_E}{dt} = \delta_1 I_H + \delta_2 A_H + \delta_3 I_C - \mu V_E$$

$$\frac{dS_C}{dt} = \Lambda_C - [\beta_H S_C(I_H + \kappa A_H) + \beta_V S_C V_E] - \mu_C S_C$$

$$\frac{dI_C}{dt} = \beta_H S_C(I_H + \kappa A_H) + \beta_V S_C V_E - (\delta_3 + \delta_C + \mu_C) I_C$$

The ten (10) compartmental model is made up of Seven (7) human compartments, two (2) vector (cats) compartments, and a compartment for the

pathogens. The model sub-divides the human population into Vaccinated individuals (V_H), Susceptible individuals (S_H), Those who are exposed to the disease (E_H) those who have been infected and can transmit the disease (I_H), the Asymptomatic people (A_H), that is those infected but don't show symptoms and can also transmit the virus, the hospitalized humans (H_H), the recovered persons (R_H), from the disease, and the reservoir (V_E), that is the environment where the disease thrives. The cats' population is also divided into susceptible cats (S_C) and infected cats (I_C).

The population of the vaccinated class increases due to the vaccination of susceptible individuals at the rate ν and decreases as a result of the waning of the efficacy of the vaccine given by ψ , and the natural death of individuals μ_H decreases the population for all human compartments.

The population of susceptible class increases due to the recruitment rate Λ_H at birth of newborns that are not vaccinated and those individuals in whom the efficacy of the vaccine has waned. And the population of the susceptible class decreases as a result of individuals getting infected, given by the function

$\Lambda_H + \psi V_H - [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\nu + \mu_H) S_H$, where β_H, β_C , or β_V are the disease transmission rates from human, cats, and the environment, respectively.

The population of exposed class (E_H) increases by a function

$[\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H$ and decreases by $(\omega_H + \mu_H) E_H$, where ω_H is the modification parameter which accounts for incubation period of the disease in the individuals infected. The population of the symptomatic infectious compartment increases when individuals leaves the exposed class by a function $(1 - \sigma)\omega_H E_H$, where $(1 - \sigma)$ is the

proportion of individuals entering the symptomatic infectious class from the exposed compartment, and the population decreases by $(\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H$, where δ_{H_1} is the disease induced death rate of the symptomatic infected individuals.

The population of asymptomatic infected individuals A_H increases by a function $\sigma\omega_H E_H$, where σ is the proportion of those that contract an infection, although they do not show symptoms, while decreases by the function $(\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H$, where δ_{H_2} is the disease induced death rate of asymptomatic infected individuals, and α_2 is the rate at which asymptomatic infected humans are hospitalized leaving the asymptomatic compartment.

The hospitalized individuals' population increases when individuals tested and found positive gets hospitalized at the rate α_1 and α_2 , from symptomatic and asymptomatic compartments, respectively, and decreases by the function $(\delta_{H_3} + \gamma + \mu_H) H_H$, where γ is the recovery rate of hospitalized individuals.

The population of susceptible cats S_C increases with the natural recruitment rate at birth of newborn cats Λ_C , and decreases when the cats comes in sufficient contact with infected humans I_H , or the exposed environment V_E . This is given by the function $\beta_H S_C (I_H + \kappa A_H) + \beta_V S_C V_E$, and the natural death rate, given by μ_C . The population of infected cats increases by the function $\beta_H S_C (I_H + \kappa A_H) + \beta_V S_C V_E$ and decreases by $(\delta_3 + \delta_C + \mu_C) I_C$.

3.0 ANALYTICAL RESULTS

In this section, we present the analytical results of the covid-19 model with vaccination and pets (cats) compartments. We established the positivity of solution of the system of equations with constant control, studied the invariant region, established the

disease free and endemic equilibrium states of the system, and determined the basic reproduction number.

$V_H > 0, S_H > 0, E_H > 0, I_H > 0, A_H > 0, H_H > 0, R_H > 0 \in \mathbb{R}_+^7$,
 $S_C > 0, I_C > 0 \in \mathbb{R}_+^2$, and $V_E > 0 \in \mathbb{R}_+$, then the solution set
 $(V_H(t), S_H(t), E_H(t), I_H(t), A_H(t), H_H(t), R_H(t))$,
 $(S_C(t), I_C(t))$, and $V_E(t)$ are positive for all time t .

Proof:

From the first equation in (1) we have

$$\frac{dV_H}{dt} = \nu S_H - (\psi + \mu_H) V_H, \text{ which can be}$$

expressed as an inequality thus;

$\frac{dV_H}{dt} \geq -(\psi + \mu_H) V_H$. Using separation of variable technique and integrating, we can have the solution as $V_H(t) \geq V_H(0)e^{-\int(\psi + \mu_H)dt}$ and conclude that $V_H(t) > 0$. Using similar approach for

$$S_H(t), E_H(t), I_H(t), A_H(t), H_H(t), R_H(t),$$

$S_C(t), I_C(t)$, and $V_E(t)$ shows that the solution is

positive \forall time t , such that the solution set

$\{V_H(t), S_H(t), E_H(t), I_H(t), A_H(t), H_H(t), R_H(t), V_E(t), S_C(t), I_C(t)\}$

is positive $\forall t$.

3.2 Invariant Region

The model equations (1) were analyzed in a biologically feasible region.

Theorem 2: The solutions to the system (1) with initial conditions in theorem 1 satisfied

$$V_H > 0, S_H > 0, E_H > 0, I_H > 0, A_H > 0, \\ H_H > 0, R_H > 0, V_E > 0, S_C > 0, I_C > 0 \\ \forall t > 0.$$

The regions $\Omega_H^* \in \mathbb{R}_+^7$, $\Omega_C^* \in \mathbb{R}_+^2$, and

$N_C(t)$, and $N_E(t)$ are also bounded. Therefore, it is epidemiologically and mathematically well-

3.1 Positivity of Solution

Theorem 1: Let the initial solution sets

$\Omega_E^* \in \mathbb{R}_+$, are positively invariant and attracting with respect to system (1). Given that

$$N_H = V_H + S_H + E_H + I_H + A_H + H_H + R_H,$$

$$N_E = V_E, \text{ and}$$

$$N_C = S_C + I_C, \text{ such that}$$

$$\Omega_H = \Omega_H^* \subset \mathbb{R}_+^7, \quad \Omega_E = \Omega_E^* \subset \mathbb{R}_+,$$

$$\Omega_C = \Omega_C^* \subset \mathbb{R}_+^2, \text{ with}$$

$$\Omega_H^* = \{(V_H, S_H, E_H, I_H, A_H, H_H, R_H) \in \mathbb{R}_+^7;$$

$$V_H + S_H + E_H + I_H + A_H + H_H + R_H \leq \frac{\Lambda_H}{\mu_H}\}$$

$$\Omega_C^* = \{(S_C, I_C) \in \mathbb{R}_+^2; S_C + I_C \leq \frac{\Lambda_C}{\mu_C}\},$$

Proof:

For human population:

$$\frac{dN_H}{dt} = \frac{dV_H}{dt} + \frac{dS_H}{dt} + \frac{dE_H}{dt} + \frac{dI_H}{dt} + \frac{dA_H}{dt} + \frac{dH_H}{dt} + \frac{dR_H}{dt}$$

$$\frac{dN_H}{dt} = \Lambda_H - \mu_H(N_H) - (\delta_{H_1} I_H + \delta_{H_2} A_H + \delta_{H_3} H_H).$$

This can be written as an inequality

$$\frac{dN_H}{dt} \leq \Lambda_H - \mu_H N_H. \text{ The inequality is}$$

expressed in a general solution as

$$N_H(t) \leq \frac{\Lambda_H}{\mu_H} - \frac{(\Lambda_H - \mu_H N_H(0))}{\mu_H} e^{-\mu_H t}, \text{ given}$$

that at initial condition $t = 0$, and

$$N_H(t) = N_H(0).$$

Therefore, as $t \rightarrow \infty$, the population size

$$N_H \rightarrow \frac{\Lambda_H}{\mu_H} \text{ implying that } N_H(t) \text{ is bounded as}$$

$$0 \leq N_H \leq \frac{\Lambda_H}{\mu_H} \text{ Similar approach proved that}$$

posed, hence it is sufficient to study the dynamic of the model in Ω (Pinto & Carvalho, 2015).

3.3 Disease free equilibrium

At disease free, we set

$E_H = I_H = A_H = H_H = I_C = 0$, and solving the equations in the system (1) we have

$$(V_H^0, S_H^0, E_H^0, I_H^0, A_H^0, H_H^0, R_H^0, V_E^0, S_C^0, I_C^0) = \left(\frac{\nu \Lambda_H}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi}, \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}, 0, 0, 0, 0, 0, 0, \frac{\Lambda_C}{\mu_C}, 0 \right)$$

3.4 Endemic equilibrium

Using substitution method to solve for equations in system (1), we obtained

$$\left. \begin{aligned} S_H^* &= \frac{\Lambda_H(\psi + \mu_H)}{[\lambda_1^* + (\nu + \mu_H)](\psi + \mu_H) - \nu \psi} \\ V_H^* &= \frac{\nu \Lambda_H}{[\lambda_1^* + (\nu + \mu_H)](\psi + \mu_H) - \nu \psi} \\ E_H^* &= \frac{\lambda_1^* \Lambda_H(\psi + \mu_H)}{(\omega_H + \mu_H) \{ [\lambda_1^* + (\nu + \mu_H)](\psi + \mu_H) - \nu \psi \}} \\ I_H^* &= \frac{\lambda_1^* \Lambda_H(1 - \sigma) \omega_H(\psi + \mu_H)}{(\omega_H + \mu_H)(\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) \{ [\lambda_1^* + (\nu + \mu_H)](\psi + \mu_H) - \nu \psi \}} \\ A_H^* &= \frac{\sigma \omega_H \lambda_1^* \Lambda_H(\psi + \mu_H)}{(\omega_H + \mu_H)(\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) \{ [\lambda_1^* + (\nu + \mu_H)](\psi + \mu_H) - \nu \psi \}} \\ H_H^* &= \frac{\alpha_1 \lambda_2^* + \alpha_2 \lambda_3^*}{(\delta_{H_3} + \gamma + \mu_H)} \\ R_H^* &= \frac{\gamma(\alpha_1 \lambda_2^* + \alpha_2 \lambda_3^*)}{\mu_H(\delta_{H_3} + \gamma + \mu_H)} \\ V_E^* &= \frac{(\delta_1 \lambda_2^* + \delta_2 \lambda_3^* + \delta_3 I_C^*)}{\mu} \\ S_C^* &= \frac{\Lambda_C - [\beta_H S_C(\lambda_2^* + \kappa \lambda_3^*) + \beta_V S_C \lambda_4^*]}{\mu_C} \\ I_C^* &= \frac{\beta_H S_C(\lambda_2^* + \kappa \lambda_3^*) + \beta_V S_C \lambda_4^*}{(\delta_3 + \delta_C + \mu_C)} \end{aligned} \right\}$$

(2)

where $\lambda_1^* = \beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E$,
 $\lambda_2^* = I_H^*$, $\lambda_3^* = A_H^*$, $V_E^* = \lambda_4^*$.

3.5 Basic Reproduction Number

The basic reproduction number, R_0 , is the average number of secondary infections generated by an infectious individual who is introduced in a wholly susceptible population. We use the next generation

matrix to determine the basic reproduction number, R_0 (Diekmann, Heesterbeek, and Metz, 1990).

$$F = \begin{bmatrix} E_H \\ I_H \\ A_H \\ H_H \\ V_E \\ I_C \end{bmatrix} = \begin{bmatrix} [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H + (\nu + \mu_H) S_H \\ 0 \\ 0 \\ 0 \\ 0 \\ [\beta_H S_C(I_H + \kappa A_H) + \beta_V S_C V_E] + \mu_C S_C \end{bmatrix},$$

$$V_i = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = -V = \begin{bmatrix} (\omega_H + \mu_H) E_H \\ (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H - (1 - \sigma) \omega_H E_H \\ (\delta_{H_1} + \alpha_2 + \delta_2 + \mu_H) A_H - \sigma \omega_H E_H \\ (\delta_{H_3} + \gamma + \mu_H) H_H - \alpha_1 I_H - \alpha_2 A_H \\ \mu V_E - \delta_1 I_H - \delta_2 A_H - \delta_3 I_C \\ (\delta_3 + \delta_C + \mu_H) I_C \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & \beta_H S_H^0 & \beta_H \kappa S_H^0 & 0 & \beta_V S_H^0 & \beta_C S_H^0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_H S_C^0 & \beta_H \kappa S_C^0 & 0 & \beta_V S_C^0 & 0 \end{bmatrix}$$

But $S_H^0 = \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}$ and

$$S_C^0 = \frac{\Lambda_C}{\mu_C} \text{ at disease free equilibrium, therefore}$$

$$F = \begin{bmatrix} 0 & q_1 & \kappa q_1 & 0 & q_3 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_4 & \kappa q_4 & 0 & q_5 & 0 \end{bmatrix}, \quad (3)$$

From (3), we let

$$q_1 = \beta_H \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi},$$

$$q_2 = \beta_C \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi},$$

$$q_3 = \beta_V \frac{\Lambda_H (\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu\psi}, \quad q_4 = \beta_H \frac{\Lambda_C}{\mu_C},$$

$$q_5 = \beta_V \frac{\Lambda_C}{\mu_C}.$$

$$V = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ -(1-\sigma)\omega_H & k_2 & 0 & 0 & 0 & 0 \\ -\sigma\omega_H & 0 & k_3 & 0 & 0 & 0 \\ 0 & -\alpha_1 & -\alpha_2 & k_4 & 0 & 0 \\ 0 & -\delta_1 & -\delta_2 & 0 & \mu & -\delta_3 \\ 0 & 0 & 0 & 0 & 0 & k_5 \end{bmatrix}, \quad (4)$$

where $k_1 = (\omega_H + \mu_H)$, $k_2 = (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H)$,
 $k_3 = (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H)$, $k_4 = (\delta_{H_3} + \gamma + \mu_H)$, and
 $k_5 = (\delta_3 + \delta_C + \mu_C)$.

$$F(V^{-1}) = \begin{bmatrix} A_1 & A_2 & A_3 & 0 & A_4 & A_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 & B_4 & B_5 \end{bmatrix} \quad (5)$$

where in (5) we let

$$A_1 = \frac{1}{k_1 k_2 q_1} (\omega_H - \sigma\omega_H) + \kappa\sigma \frac{\omega_H}{k_1 k_3 q_1} + \frac{1}{\mu k_1 k_2 k_3 q_3} (\delta_1 \omega_H k_3 - \sigma\delta_1 \omega_H k_3 + \sigma\delta_2 \omega_H k_3),$$

$$A_2 = \frac{1}{k_2 q_1} + \frac{\delta_1}{\mu k_2 q_3}, \quad A_3 = \frac{\kappa}{k_3 q_1} + \frac{\delta_2}{\mu k_3 q_3}, \quad A_4 = \frac{1}{\mu q_3},$$

$$A_5 = \frac{1}{k_5 q_2} + \frac{\delta_3}{\mu k_5 q_3},$$

$$B_1 = \frac{1}{k_1 k_2 q_4} (\omega_H - \sigma\omega_H) + \kappa\sigma \frac{\omega_H}{k_1 k_3 q_4} + \frac{1}{\mu k_1 k_2 k_3 q_5} (\delta_1 \omega_H k_3 - \sigma\delta_1 \omega_H k_3 + \sigma\delta_2 \omega_H k_2),$$

$$B_2 = \frac{1}{k_2 q_4} + \frac{\delta_1}{\mu k_2 q_5}, \quad B_3 = \frac{\kappa}{k_3 q_4} + \frac{\delta_2}{\mu k_3 q_5}, \quad B_4 = \frac{1}{\mu q_5},$$

$$B_5 = \frac{\delta_3}{\mu k_5 q_5}.$$

Therefore, we evaluate the characteristic equation
 $|FV^{-1} - \lambda I| = 0$ and our basic reproduction
 number is therefore

$$R_0 = \frac{1}{2} \left(A_1 + B_5 + \sqrt{(A_1 - B_5)^2 + 4B_1 A_5} \right) \quad (6)$$

4.0 STABILITY ANALYSIS

We examine the stability of the model system for covid-19 disease at disease free equilibrium point.

4.1 Local stability analysis

Theorem 3: The disease free equilibrium E^0 point of the model is locally asymptotically stable (LAS) if $R_0 < 1$, and unstable if otherwise.

Proof: We let

$$\left. \begin{aligned} F_1 &= \nu S_H - (\psi + \mu_H) V_H \\ F_2 &= \Lambda_H + \psi V_H - [\beta_H (I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\nu + \mu_H) S_H \\ F_3 &= [\beta_H (I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\omega_H + \mu_H) E_H \\ F_4 &= (1 - \sigma) \omega_H E_H - (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H \\ F_5 &= \sigma \omega_H E_H - (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H \\ F_6 &= \alpha_1 I_H + \alpha_2 A_H - (\delta_{H_3} + \gamma + \mu_H) H_H \\ F_7 &= \gamma H_H - \mu_H R_H \\ F_8 &= \delta_1 I_H + \delta_2 A_H + \delta_3 I_C - \mu V_E \\ F_9 &= \Lambda_C - [\beta_H S_C (I_H + \kappa A_H) + \beta_V S_C V_E] - \mu_C S_C \\ F_{10} &= \beta_H S_C (I_H + \kappa A_H) + \beta_V S_C V_E - (\delta_3 + \delta_C + \mu_C) I_C \end{aligned} \right\} \quad (7)$$

At disease free equilibrium, that is

$$\begin{aligned}
& (V_H^0, S_H^0, E_H^0, I_H^0, A_H^0, H_H^0, R_H^0, V_E^0, S_C^0, I_H^0) = \\
& \left(\frac{\nu \Lambda_H}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi}, \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}, 0, 0, 0, 0, 0, \frac{\Lambda_C}{\mu_C}, 0 \right) \\
& J_{E^0} = \begin{vmatrix} -r_1 & \nu & 0 & 0 & 0 & 0 & 0 \\ \psi & -r_2 & 0 & -b_1 & -b_2 & -b_3 & -b_4 \\ 0 & 0 & -r_3 & b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & (1-\sigma)\omega_H & -r_4 & 0 & 0 & 0 \\ 0 & 0 & \sigma\omega_H & 0 & -r_5 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -r_6 & \delta_3 \\ 0 & 0 & 0 & c_{74} & c_{75} & c_{76} & -r_7 \end{vmatrix} = 0 \quad (8)
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \beta_H \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi} \\
b_2 &= \beta_H \kappa \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}, \\
b_3 &= \beta_V \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}, \\
c_{74} &= \beta_H \frac{\Lambda_C}{\mu_C}, c_{75} = \beta_H \kappa \frac{\Lambda_C}{\mu_C}, c_{76} = \beta_V \frac{\Lambda_C}{\mu_C}, \\
r_1 &= (\psi + \mu_H), r_2 = (\nu + \mu_H), \\
r_3 &= (\omega_H + \mu_H), r_4 = (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H), \\
r_5 &= (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H), \\
r_6 &= \mu, r_7 = (\delta_3 + \delta_C + \mu_C),
\end{aligned}$$

So for $|J_{E^0} - \lambda I| = 0$, we obtained the characteristic polynomial as

$$D_1 \lambda^7 + D_2 \lambda^6 + D_3 \lambda^5 + D_4 \lambda^4 + D_5 \lambda^3 + D_6 \lambda^2 + D_7 \lambda + D_0 = 0 \quad (9)$$

We apply Routh-Hurwitz criterion according to which all roots of the polynomial (9) have negative real parts if and only if the coefficients D_i , are positive and the determinant of the matrices $H_i > 0$ for $i = 0, 1, 2, 3, 4, 5, 6, 7$.

Lemma 1: The point $K^0 = (X^0, 0)$ is referred to be the stable global asymptotic equilibrium point if additionally, $R_0 < 1$ and the conditions H_1 & H_2 holds true. The resulting theorem is formed:

Hence, the polynomial (9) have negative real parts in all of its eigenvalues. Given that all values of $\lambda_i < 0$, for $i = 1, 2, 3, \dots, 10$ when $R_0 < 1$, we conclude that the disease-free equilibrium point is locally asymptotically stable (see Appendix II for the analysis of the Routh-Hurwitz criterion).

4.2 Global stability analysis

To determine the global stability of the disease-free equilibrium, we applied Castillo-Chavez *et al.* (2002) methodology. Thus, our system of equations (1) is rewritten as follows

$$\left. \begin{aligned} \frac{dX}{dt} &= F(X, Z) \\ \frac{dZ}{dt} &= G(X, Z), G(X, 0) = 0 \end{aligned} \right\} \quad (10)$$

where $X = (V_H, S_H, R_H, S_C)$ denotes the number of uninfected individuals and $X \in R^4$, while $Z = (E_H, I_H, A_H, H_H, V_E, I_C)$ denotes the number of infected individuals and $Z \in R^6$. We represent the disease-free state by $E^0 = (X^0, 0)$

.The two conditions (H_1 & H_2)

must be met to guarantee a global asymptotic stability:

$$\begin{aligned}
H_1 : & \text{for } \frac{dX}{dt} = F(X, 0), X^0 \text{ is globally} \\
& \text{asymptotically stable} \quad (11)
\end{aligned}$$

$$H_2 : G(X, Z) = CZ - G(X, Z), \text{ where}$$

$$G(X, Z) \geq 0, \text{ for } (X, Z) \in \Omega \quad (12)$$

where $C = D_Z G(X^0, 0)$ is an M -matrix (the off diagonal of C are non-negative) and Ω is the biological feasible region.

Theorem 4: Let $R_0 < 1$, the disease-free equilibrium is then globally asymptotically stable.
Proof:

Let $X = (V_H, S_H, R_H, S_C)$,
 $Z = (E_H, I_H, A_H, H_H, V_E, I_C)$ and
 $K^0 = (X^0, 0)$ where

$$X^0 = \left(\frac{\nu \Lambda_H}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi}, \frac{\Lambda_H(\psi + \mu_H)}{(\nu + \mu_H)(\psi + \mu_H) - \nu \psi}, 0, \frac{\Lambda_C}{\mu_C} \right) \quad (13)$$

$$\left. \begin{aligned} X \in R^4 \Rightarrow \\ \frac{dV_H}{dt} &= \nu S_H - (\psi + \mu_H) V_H \\ \frac{dS_H}{dt} &= \Lambda_H + \psi V_H - [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\nu + \mu_H) S_H \\ \frac{dR_H}{dt} &= \gamma H_H - \mu_H R_H \\ \frac{dS_C}{dt} &= \Lambda_C - [\beta_H S_C(I_H + \kappa A_H) + \beta_V S_C V_E] - \mu_C S_C \end{aligned} \right\} \quad (14)$$

$$F(X, 0) = \begin{pmatrix} \frac{\nu \Lambda_H}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi} \\ \frac{\Lambda_H(\psi + \mu_H)}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi} \\ 0 \\ \frac{\Lambda_C}{\mu_C} \end{pmatrix} \quad (15)$$

$$\left. \begin{aligned} Z \in R^6 \Rightarrow \\ \frac{dE_H}{dt} &= [\beta_H(I_H + \kappa A_H) + \beta_C I_C + \beta_V V_E] S_H - (\omega_H + \mu_H) E_H \\ \frac{dI_H}{dt} &= (1 - \sigma) \omega_H E_H - (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H \\ \frac{dA_H}{dt} &= \sigma \omega_H E_H - (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H \\ \frac{dH_H}{dt} &= \alpha_1 I_H + \alpha_2 A_H - (\delta_{H_3} + \gamma + \mu_H) H_H \\ \frac{dV_E}{dt} &= \delta_1 I_H + \delta_2 A_H + \delta_3 I_C - \mu V_E \\ \frac{dI_C}{dt} &= \beta_H S_C(I_H + \kappa A_H) + \beta_V S_C V_E - (\delta_3 + \delta_C + \mu_C) I_C \end{aligned} \right\} \quad (16)$$

$$C = \begin{bmatrix} -(\omega_H + \mu_H) & \beta_H S_H^0 & \beta_H \kappa S_H^0 & 0 & \beta_V S_H^0 & \beta_C S_H^0 \\ (1 - \sigma) \omega_H & -(\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) & 0 & 0 & 0 & 0 \\ \sigma \omega_H & 0 & -(\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & -(\delta_{H_3} + \gamma + \mu_H) & 0 & 0 \\ 0 & \delta_1 & \delta_2 & 0 & -\mu & \delta_3 \\ 0 & \beta_H S_C^0 & \beta_H \kappa S_C^0 & 0 & \beta_V S_C^0 & -(\delta_3 + \delta_C + \mu_C) \end{bmatrix} \quad (17)$$

$$\hat{G}(X, Z) = \begin{pmatrix} \hat{G}_1(X, Z) \\ \hat{G}_2(X, Z) \\ \hat{G}_3(X, Z) \\ \hat{G}_4(X, Z) \\ \hat{G}_5(X, Z) \\ \hat{G}_6(X, Z) \end{pmatrix} = \begin{bmatrix} [\beta_H I_H + \beta_H \kappa A_H + \beta_V V_E + \beta_C I_C] (S_H^0 - S_H) \\ 0 \\ 0 \\ 0 \\ 0 \\ [\beta_H I_H + \beta_H \kappa A_H + \beta_V V_E] (S_C^0 - S_C) \end{bmatrix} \quad (18)$$

Therefore, since $S_H \leq S_H^0$, and $S_C \leq S_C^0$, the

population is bounded and $\hat{G}(X, Z) \geq 0$.

The global stability of $X^0 = \left(\frac{\nu \Lambda_H}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi}, \frac{\Lambda_H(\psi + \mu_H)}{(\psi + \mu_H)(\nu + \mu_H) - \nu \psi}, 0, \frac{\Lambda_C}{\mu_C} \right)$

then system $\frac{dX}{dt} = F(X^0, 0)$ is easy to verify.

Therefore, X^0 is globally asymptotically stable if $R_0 < 1$. This completes the proof.

5. OPTIMAL CONTROL ANALYSIS

We investigate the optimal level of efforts that will be needed in controlling the disease transmission.

Therefore, we examine the objective functional formulated as:

$$J(u_1, u_2, u_3, u_4) = \int_{T_0}^{T_f} \left[(P_1 E_H + P_2 I_H + P_3 A_H + P_4 H_H) + \frac{1}{2} (Q_i u_i^2) \right] dt, \quad i = 1, 2, \dots, 4. \quad (19)$$

Since our aim is to minimize the number of infections in the exposed compartment E_H , symptomatic infected compartment I_H , asymptomatic infected compartment A_H , and hospitalized individuals compartment H_H at a very low cost of control $u_1(t), u_2(t), u_3(t), u_4(t)$, we seek an optimal control $u_1^*, u_2^*, u_3^*, u_4^*$ such that

$$J(u_1^*, u_2^*, u_3^*, u_4^*) = \min_{u_1, u_2, u_3, u_4} \{ J(u_1, u_2, u_3, u_4) \mid u_1, u_2, u_3, u_4 \in U \}$$

where U is the set of measurable functions defined from $[T_0, T_f]$ onto $[0, 1]$ and P_i and Q_i are positive weight constants. Optimal control approach in biological problems is explained in Lenhart and Workman (2007). Given the system of equations in (20),

$$\left. \begin{aligned} \frac{dV_H}{dt} &= u_1 \psi S_H - (\psi + \mu_H) V_H \\ \frac{dS_H}{dt} &= \Lambda_H + \psi V_H - (1 - u_2) S_H [\beta_H (I_H + \kappa A_H) + \beta_C I_C] - [(1 - u_3) \beta_V V_E + u_1 \psi + \mu_H] S_H \\ \frac{dE_H}{dt} &= (1 - u_2) [\beta_H (I_H + \kappa A_H) + \beta_C I_C] S_H + (1 - u_3) \beta_V S_H V_E - (\omega_H + \mu_H) E_H \\ \frac{dI_H}{dt} &= (1 - \sigma) \omega_H E_H - (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H \\ \frac{dA_H}{dt} &= \sigma \omega_H E_H - (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H \\ \frac{dH_H}{dt} &= \alpha_1 I_H + \alpha_2 A_H - (\delta_{H_3} + u_4 \gamma + \mu_H) H_H \\ \frac{dR_H}{dt} &= u_4 \gamma H_H - \mu_H R_H \\ \frac{dV_E}{dt} &= \delta_1 I_H + \delta_2 A_H + \delta_3 I_C - \mu V_E \\ \frac{dS_C}{dt} &= \Lambda_C - [(1 - u_2) \beta_H S_C (I_H + \kappa A_H) + (1 - u_3) \beta_V S_C V_E] - \mu_C S_C \\ \frac{dI_C}{dt} &= (1 - u_2) \beta_H S_C (I_H + \kappa A_H) + (1 - u_3) \beta_V S_C V_E - (\delta_3 + \delta_C + \mu_C) I_C \end{aligned} \right\} \quad (20)$$

We pose the following theorem:

Theorem 5: There exists an optimal control set u_1, u_2, u_3, u_4 and the corresponding state system

(1) that minimizes $J(u_1, u_2, u_3, u_4)$ over U .

Moreover, there exists adjoint functions

$\lambda_{V_H}, \lambda_{S_H}, \lambda_{E_H}, \lambda_{I_H}, \lambda_{A_H}, \lambda_{H_H}, \lambda_{R_H}, \lambda_{V_E}, \lambda_{S_C}$, and λ_{I_C} such that our Hamiltonian is

$$\begin{aligned}
H = & P_1 E_H + P_2 I_H + P_3 A_H + P_4 H_H + \frac{1}{2} (Q_1 u_1^2 + Q_2 u_2^2 + Q_3 u_3^2 + Q_4 u_4^2) \\
& + \lambda_{V_H} [u_1 \psi S_H - (\psi + \mu_H) V_H] \\
& + \lambda_{S_H} \{ \Lambda_H + \psi V_H - (1 - u_2) S_H [\beta_H (I_H + \kappa A_H) + \beta_c I_c] - [(1 - u_3) \beta_v V_E + u_1 \psi + \mu_H] S_H \} \\
& + \lambda_{E_H} [(1 - u_2) [\beta_H (I_H + \kappa A_H) + \beta_c I_c] S_H + (1 - u_3) \beta_v S_H V_E - (\omega_H + \mu_H) E_H] \\
& + \lambda_{I_H} [(1 - \sigma) \omega_H E_H - (\delta_{H_1} + \alpha_1 + \delta_1 + \mu_H) I_H] \\
& + \lambda_{A_H} [\sigma \omega_H E_H - (\delta_{H_2} + \alpha_2 + \delta_2 + \mu_H) A_H] \\
& + \lambda_{H_H} [\alpha_1 I_H + \alpha_2 A_H - (\delta_{H_3} + u_4 \gamma + \mu_H) H_H] \\
& + \lambda_{R_H} [u_4 \gamma H_H - \mu_H R_H] \\
& + \lambda_{V_E} [\delta_1 I_H + \delta_2 A_H + \delta_3 I_c - \mu V_E] \\
& + \lambda_{S_c} [\Lambda_c - [(1 - u_2) \beta_H S_c (I_H + \kappa A_H) + (1 - u_3) \beta_v S_c V_E] - \mu_c S_c] \\
& + \lambda_{I_c} [(1 - u_2) \beta_H S_c (I_H + \kappa A_H) + (1 - u_3) \beta_v S_c V_E - (\delta_3 + \delta_c + \mu_c) I_c]
\end{aligned} \tag{21}$$

where $\lambda_{V_H}, \lambda_{S_H}, \lambda_{E_H}, \lambda_{I_H}, \lambda_{A_H}, \lambda_{H_H}, \lambda_{R_H}, \lambda_{V_E}, \lambda_{S_c}$, and λ_{I_c} are the adjoint variables or the co-state variables.

Given the transversality conditions:

$$\lambda_{V_H}(t_f) = \lambda_{S_H}(t_f) = \lambda_{E_H}(t_f) = \lambda_{I_H}(t_f) = \lambda_{A_H}(t_f) = \lambda_{H_H}(t_f) = \lambda_{R_H}(t_f) = \lambda_{V_E}(t_f) = \lambda_{S_c}(t_f) = \lambda_{I_c}(t_f) = 0 \tag{22}$$

And the controls $u_1^*, u_2^*, u_3^*, u_4^*$ satisfy the optimality conditions

$$\begin{aligned}
u_1^* = & \max \left\{ 0, \min \left(1, \frac{(\lambda_{S_H} - \lambda_{V_H}) \psi S_H}{Q_1} \right) \right\} \\
u_2^* = & \max \left\{ 0, \min \left(1, \frac{(\lambda_{E_H} - \lambda_{S_H}) \beta_H S_H (I_H + \kappa A_H) + (\lambda_{E_H} - \lambda_{S_H}) \beta_c I_c + (\lambda_{I_c} - \lambda_{S_c}) \beta_H S_c (I_H + \kappa A_H)}{Q_2} \right) \right\} \\
u_3^* = & \max \left\{ 0, \min \left(1, \frac{(\lambda_{E_H} - \lambda_{S_H}) \beta_v S_H V_E + (\lambda_{I_c} - \lambda_{S_c}) \beta_v S_c V_E}{Q_3} \right) \right\} \\
u_4^* = & \max \left\{ 0, \min \left(1, \frac{(\lambda_{H_H} - \lambda_{R_H}) \gamma H_H}{Q_4} \right) \right\}
\end{aligned} \tag{23}$$

Proof: The adjoint system were obtained by differentiating the Hamiltonian function in (21), evaluated at the optimal control which led to;

$$\begin{aligned}
\frac{d\lambda_{V_H}}{dt} &= -\frac{\partial H}{\partial V_H}, \quad \frac{d\lambda_{S_H}}{dt} = -\frac{\partial H}{\partial S_H}, \quad \frac{d\lambda_{E_H}}{dt} = -\frac{\partial H}{\partial E_H}, \\
\frac{d\lambda_{I_H}}{dt} &= -\frac{\partial H}{\partial I_H}, \quad \frac{d\lambda_{A_H}}{dt} = -\frac{\partial H}{\partial A_H}, \quad \frac{d\lambda_{H_H}}{dt} = -\frac{\partial H}{\partial H_H}, \quad (24) \\
\frac{d\lambda_{R_H}}{dt} &= -\frac{\partial H}{\partial R_H}, \quad \frac{d\lambda_{V_E}}{dt} = -\frac{\partial H}{\partial V_E}, \quad \frac{d\lambda_{S_C}}{dt} = -\frac{\partial H}{\partial S_C}, \\
\frac{d\lambda_{I_C}}{dt} &= -\frac{\partial H}{\partial I_C}
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_{V_H}}{dt} &= (\lambda_{V_H} - \lambda_{S_H})\psi + \lambda_{V_H}\mu_H \\
\frac{d\lambda_{S_H}}{dt} &= (\lambda_{S_H} - \lambda_{E_H})(1-u_2)\beta_H(I_H + \kappa A_H) + (\lambda_{S_H} - \lambda_{E_H})(1-u_2)\beta_C I_C + (\lambda_{S_H} - \lambda_{E_H})(1-u_3)\beta_V V_E + (\lambda_{S_H} - \lambda_{V_H})u_1\nu + \lambda_{S_H}\mu_H \\
\frac{d\lambda_{E_H}}{dt} &= (\lambda_{E_H} - \lambda_{I_H})\omega_H + (\lambda_{I_H} - \lambda_{A_H})\sigma\omega_H + \lambda_{E_H}\mu_H - P_1 \\
\frac{d\lambda_{I_H}}{dt} &= (\lambda_{S_H} - \lambda_{E_H})(1-u_2)\beta_H S_H + (\lambda_{S_C} - \lambda_{I_C})(1-u_2)\beta_H S_C + (\lambda_{I_H} - \lambda_{H_H})\alpha_1 + (\lambda_{I_H} - \lambda_{V_E})\delta_1 + (\delta_{H_1} + \mu_H)\lambda_{I_H} - P_2 \\
\frac{d\lambda_{A_H}}{dt} &= (\lambda_{S_H} - \lambda_{E_H})(1-u_2)\beta_H \kappa S_H + (\lambda_{S_C} - \lambda_{I_C})(1-u_2)\beta_H \kappa S_C + (\lambda_{A_H} - \lambda_{H_H})\alpha_2 + (\lambda_{A_H} - \lambda_{V_E})\delta_2 + \lambda_{A_H}(\delta_{H_2} + \mu_H) - P_3 \\
\frac{d\lambda_{H_H}}{dt} &= (\lambda_{H_H} - \lambda_{R_H})u_4\gamma + \lambda_{H_H}(\delta_{H_3} + \mu_H) - P_4 \\
\frac{d\lambda_{R_H}}{dt} &= \lambda_{R_H}\mu_H \\
\frac{d\lambda_{V_E}}{dt} &= (\lambda_{S_H} - \lambda_{E_H})(1-u_3)\beta_V S_H + (\lambda_{S_C} - \lambda_{I_C})(1-u_3)\beta_V S_C + \lambda_{V_E}\mu \\
\frac{d\lambda_{S_C}}{dt} &= (\lambda_{S_C} - \lambda_{I_C})(1-u_2)\beta_H(I_H + \kappa A_H) + (\lambda_{S_C} - \lambda_{I_C})(1-u_3)\beta_V V_E + \lambda_{S_C}\mu_C \\
\frac{d\lambda_{I_C}}{dt} &= (\lambda_{S_H} - \lambda_{E_H})(1-u_2)\beta_C S_H + (\lambda_{I_C} - \lambda_{V_E})\delta_3 + \lambda_{I_C}(\delta_C + \mu_C)
\end{aligned} \quad (25)$$

with transversality condition (22), we obtain the characterization of the controls by solving for

$u_i, (i=1,2,3,4)$ in (26)

$$\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0, \quad \frac{\partial H}{\partial u_3} = 0, \quad \text{and} \quad \frac{\partial H}{\partial u_4} = 0 \quad (26)$$

Thus

$$\begin{aligned}
u_1^* &= \frac{(\lambda_{S_H} - \lambda_{V_H})\nu S_H}{Q_1} \\
u_2^* &= \frac{(\lambda_{E_H} - \lambda_{S_H})[\beta_H S_H(I_H + \kappa A_H) + \beta_C I_C] + (\lambda_{I_C} - \lambda_{S_C})\beta_H S_C(I_H + \kappa A_H)}{Q_2} \\
u_3^* &= \frac{(\lambda_{E_H} - \lambda_{S_H})\beta_V S_H V_E + (\lambda_{S_C} - \lambda_{I_C})\beta_V S_C V_E}{Q_3} \\
u_4^* &= \frac{(\lambda_{H_H} - \lambda_{R_H})\gamma H_H}{Q_4}
\end{aligned} \quad (27)$$

Therefore, based on the standard control arguments involving the bound on the controls, we conclude that

$$\begin{aligned} u_1^* &= \begin{cases} 0 & \text{if } u_1^d \leq 0 \\ u_1^d & \text{if } 0 < u_1^d < 1 \\ 1 & \text{if } u_1^d \geq 1 \end{cases}, u_2^* = \begin{cases} 0 & \text{if } u_2^d \leq 0 \\ u_2^d & \text{if } 0 < u_2^d < 1 \\ 1 & \text{if } u_2^d \geq 1 \end{cases} \\ u_3^* &= \begin{cases} 0 & \text{if } u_3^d \leq 0 \\ u_3^d & \text{if } 0 < u_3^d < 1 \\ 1 & \text{if } u_3^d \geq 1 \end{cases}, u_4^* = \begin{cases} 0 & \text{if } u_4^d \leq 0 \\ u_4^d & \text{if } 0 < u_4^d < 1 \\ 1 & \text{if } u_4^d \geq 1 \end{cases} \end{aligned} \quad (28)$$

Due to the apriori boundedness of the solutions to both the state and adjoint equations., we determine the uniqueness of the optimality system (21) – (23).

The restriction of time interval duration $[0, t_f]$ to ensure the uniqueness of the optimality system. This is because the state problem has starting values whereas the adjoint problem has final values due to the opposite temporal orientations of (21) - (23). In control problems, this constraint is typical, (Alhassan, Momoh, Abdullahi & Kadzai, 2017).

6. NUMERICAL SIMULATIONS

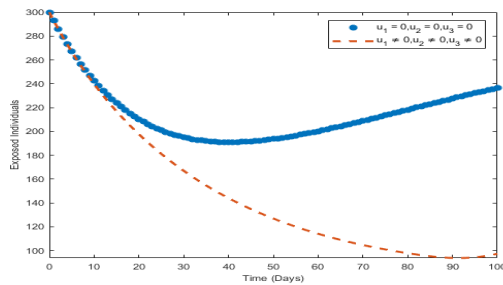
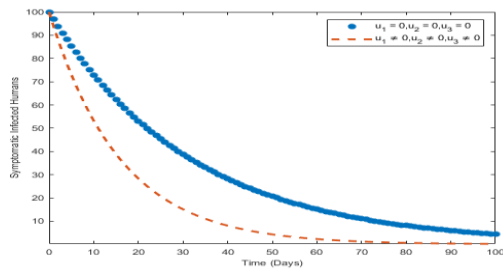
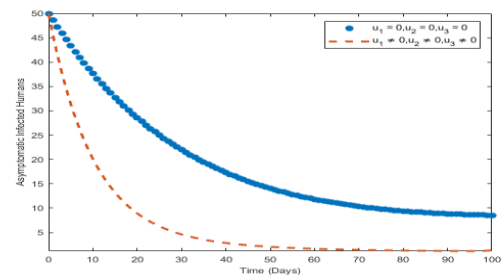
In this section, we use values of the parameter and state initial values from Tables 1 and 2 respectively, to investigate numerically, how the best control measures might affect the spread of Covid-19 in a population, which we implemented in Matlab.

Table 1: Parameters and Their Values

Parameters	Range	Value	Units	Source
Λ_H		0.01888		World Bank
Λ_C		0.0351		Assumed
μ_H		0.00712		World Bank
μ_C		0.00032		Assumed
μ		0.1724	Day ⁻¹	Mwalili, (2020)
δ_{H_1}		3.5	Day ⁻¹	Ndairou, (2020)
δ_{H_2}		1	Day ⁻¹	Ndairou, (2020)
δ_{H_3}		0.3	Day ⁻¹	Ndairou, (2020)
δ_C		0.00003	Day ⁻¹	Assumed
ν		0.35		Assumed
β_H		0.0805	Day ⁻¹	Ahmed, (2021)
β_V		0.10		Mwalili, (2020)
β_C		0.61		Assumed
α_1	0.2 – 0.5	0.33604	Day ⁻¹	Olaniyi, (2020)
α_2	0.1 – 0.25	0.19466	Day ⁻¹	Olaniyi, (2020)
\mathcal{S}		0.1	Day ⁻¹	Mwalili, (2020)
δ_2		0.05	Day ⁻¹	Mwalili, (2020)
δ_3		0.15	Day ⁻¹	Assumed
ψ		95%		Choi, (2021)
γ		0.06666667	Day ⁻¹	Olaniyi, (2020)
$(1 - \sigma)$		0.5		Olaniyi, (2020)
σ		0.5		Olaniyi, (2020)
ω_H		0.143	Day ⁻¹	Deressa (2020)
κ		0.3		Khoshnaw (2020)
P_1		1		Assumed
P_2		1		Olaniyi, (2020)
P_3		1		Olaniyi, (2020)
P_4		1		Olaniyi, (2020)
Q_1		20		Assumed
Q_2		45		Sasmita (2020)
Q_3		40		Assumed
Q_4		90		Sasmita (2020)

Table 2: State Variables at Initial Values Used in the Model

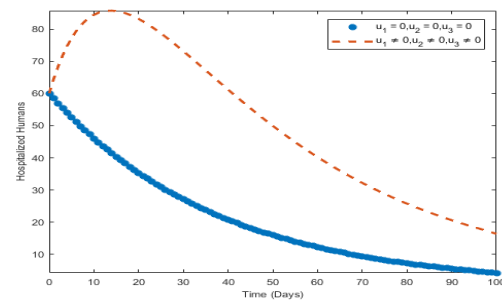
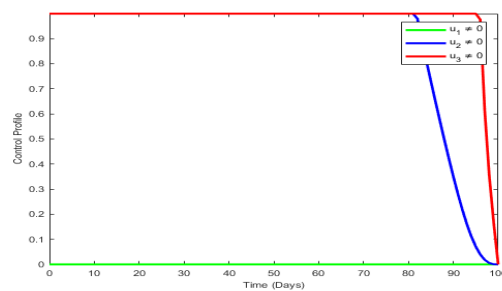
State variables	Values
$V_H(0)$	500
$S_H(0)$	1000
$E_H(0)$	300
$I_H(0)$	100
$A_H(0)$	50
$H_H(0)$	70
$R_H(0)$	60
$V_E(0)$	5000
$S_C(0)$	1500
$I_C(0)$	200

**Fig. 1(a)****Fig. 1(b)**

The strategies implemented are:

- Strategy I: Use of vaccination, Physical distancing and face mask, and sanitation for the population.
- Strategy II: Use of vaccination, physical distancing and face mask, and treatment for the susceptible as well as infected population.
- Strategy III: Use of vaccination, sanitation, and treatment for the susceptible and infected population.
- Strategy IV: Use of face mask and physical distancing, sanitation, and treatment for the susceptible and infected population
- Strategy V: Use of vaccination, face mask and physical distancing, and treatment for the whole population.

Figures 1 to 5 show the outcomes of the numerical simulations.

Fig. 1(c)**Fig. 1(d)****Fig. 1(e)****Figure 1: Implementation of strategy I.**

6.1. Strategy I: an optimal use of vaccination, physical distancing and face mask, and sanitation as control strategy

Strategy I, vaccination, physical distancing and face mask, and sanitation, (u_1, u_2, u_3) , is used to mitigate the spread of covid-19 virus amongst susceptible population by optimizing the objective functional (J), setting u_4 to zero. Figures 4.1a – 4.1d shows some differences in the population of individuals who contracted the covid-19 virus with control measures and without. Figure 1(a) shows a significant difference in the exposed human population E_H with optimal control strategy compared to E_H without control. We observed in figures 1(a), 1(b), and 1(c) that as a result of the control technique, the number of exposed humans E_H , infected individuals I_H , asymptomatic infected individuals A_H reduces, while when there is no control, the population grows. In figure 1(d) however, it is not so with population of hospitalized humans. This is to say that control strategy I is not favorable to hospitalized humans. The control profile in figure 1(e), revealed that it requires 100% implementation of use of face mask and physical distancing, and sanitation for 80 days and 96 days respectively before they dropped to their lower bounds, and for vaccination it stays at lower bound.

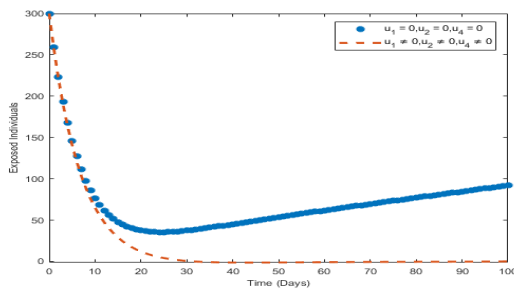


Fig. 2(a)

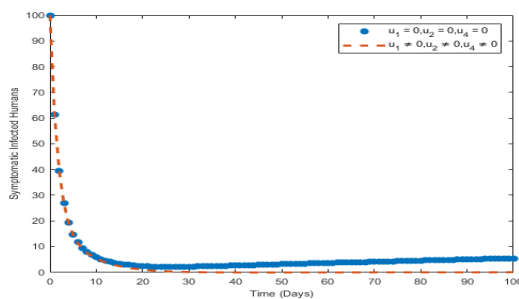


Fig. 2(b)

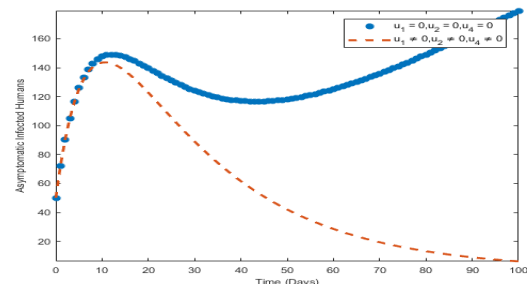


Fig. 2(c)

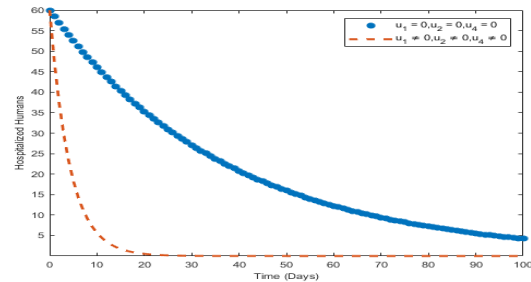


Fig. 2(d)

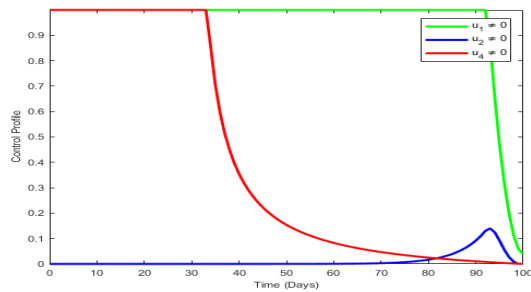


Fig. 2(e)

Figure 2: Implementation of strategy II.

6.2 Strategy II: an optimal use of vaccination, face mask and physical distancing, and treatment as control strategy

In this strategy, the optimal use of vaccines, use of face mask and physical distancing, and treatment (u_1, u_2, u_4) were used as the only control strategy to optimize the objective functional (J), setting $u_3 = 0$. Thus, it can be observed in figures 2(a) to 2(d), that the number of exposed humans E_H , infected individuals I_H , asymptomatic infected individuals A_H , and hospitalized humans, decreases due to the control strategy, while the incidence increases when there is no control. The control profile depicted in figure 2(e) showed that

administration of vaccines, use of face mask and practicing of physical distancing, and treatment as control strategy for mitigating the transmission of covid-19 virus is at the upper bound at 100% implementation of u_1 and u_4 for 32 days and 93 days respectively, and use of face mask and physical distancing for 93 days before they dropped down to their lower bounds

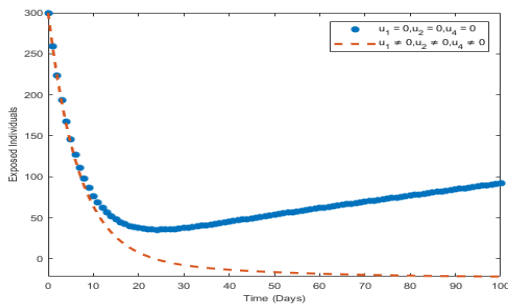


Fig. 3(a)

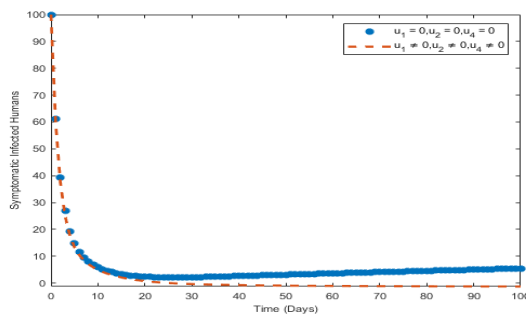


Fig. 3(b)

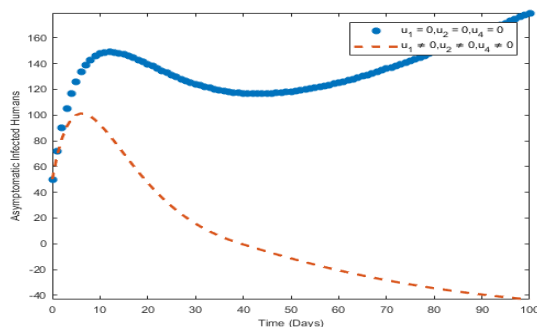


Fig. 3(c)

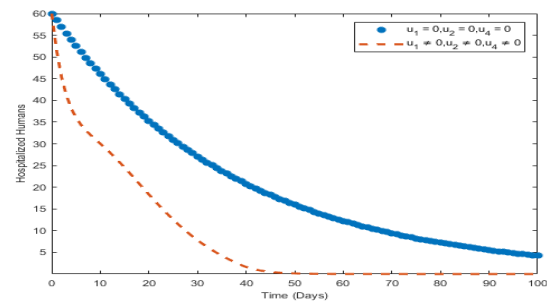


Fig. 3(d)

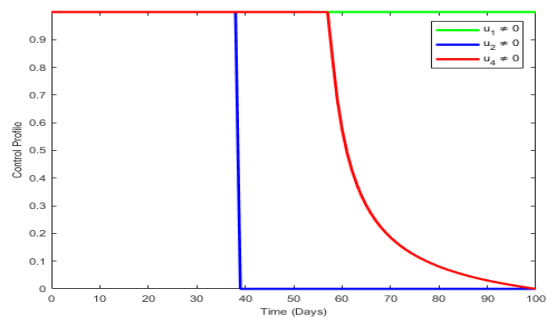


Fig. 3(e)

Figure 3: Implementation of strategy III

6.3 Strategy III: an optimal use of vaccination, sanitation, and treatment as control strategy

Optimal use of vaccines, sanitation, and treatment: considering the administration of vaccines, sanitation, and treatment (u_1, u_3, u_4) , we set the

use of face masks and physical distancing (u_2) to zero in the susceptible human population as the only control strategy to optimize the functional (J). Figures 3(a), 3(b), and 3(c) shows that in the presence of control, the number of exposed individuals E_H , infected people I_H , asymptomatic infected individuals A_H , and hospitalized individuals H_H decreases to the minimal level as compared with the number without control. The control profile in figure 3(d), shows an upper bound at 100% implementation of the three controls of vaccination, sanitation, and treatment in 39 days, 60 days, and 100 days respectively before they dropped down to lower bound.

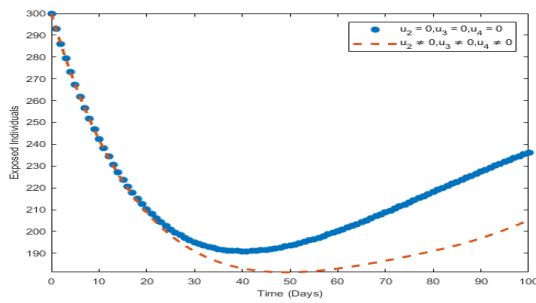


Fig. 4(a)

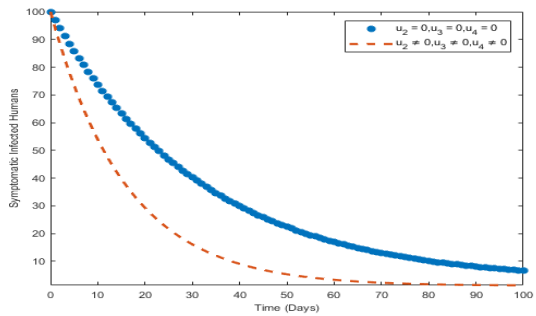


Fig. 4(b)

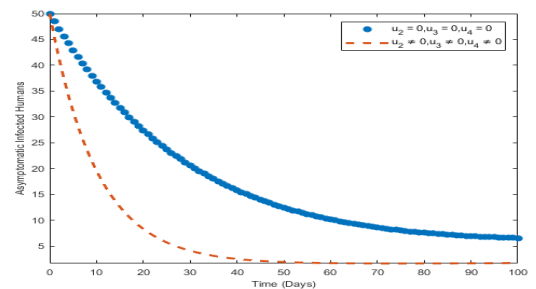


Fig. 4(c)

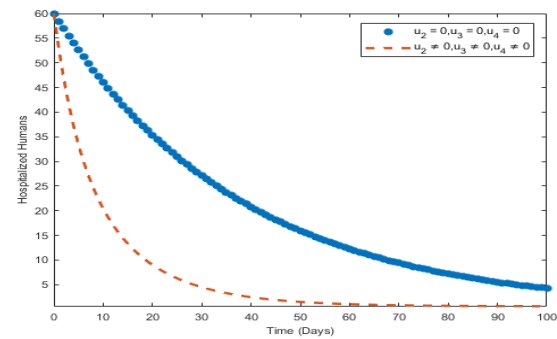


Fig. 4(d)

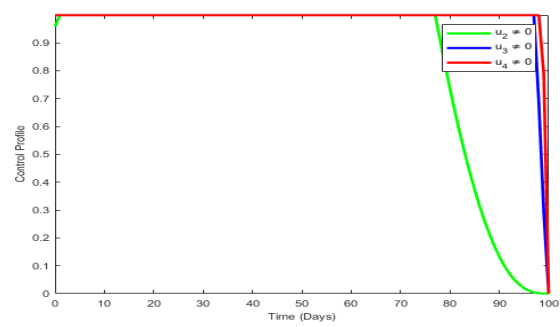


Fig. 4(e)

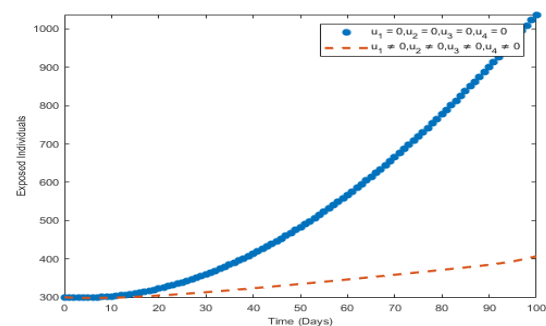
Figure 4: Implementation of strategy IV

6.4 Strategy IV: an optimal use of face mask and physical distancing, sanitation, and treatment as control strategy

Optimal use of face mask and physical distancing, sanitation, and treatment: we used the control of use of face mask and physical distancing u_2 , sanitation

u_3 , and treatment u_4 , which were combined to optimize the objective functional (J), while setting the vaccination u_1 to zero. In figure 4(a) to 4(d), the population of exposed humans E_H , symptomatically infected I_H , asymptotically infected A_H , and hospitalized individuals H_H

decreases in the presence of controls u_2, u_3, u_4 , meanwhile the number of individuals in those classes increases when there is no optimal strategy. Meanwhile, the control profile in figure 4(e), shows that the controls u_2, u_3, u_4 are at upper bound 100% implementation of the three controls in 80 days, 97 days, and 98 days respectively before dropping down to lower bound.



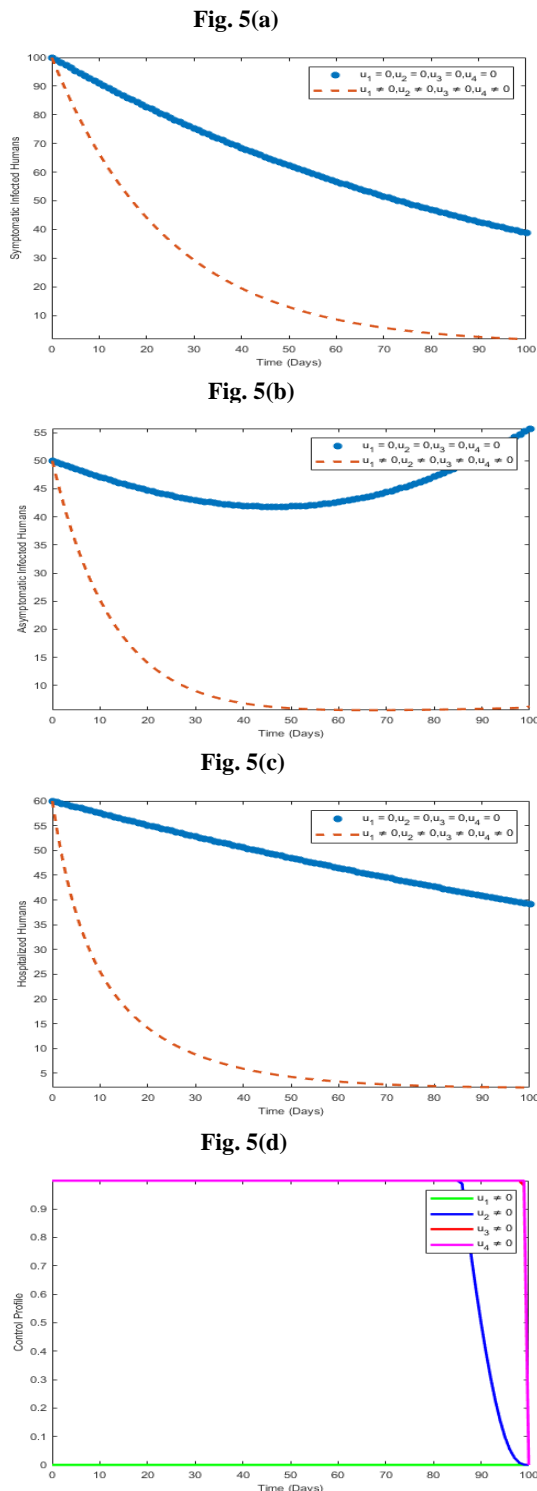


Figure 5: Implementation of strategy V
6.1.5 Strategy V: an optimal use of vaccination, face mask and physical distancing, sanitation, and treatment as control strategy

Optimal use of vaccination, face mask and physical distancing, sanitation, and treatment u_1, u_2, u_3, u_4 : in this strategy, the four controls are used to optimize the objective functional (J). Figure 5(a) to 5(d) shows a significant difference in the infected population with optimal control strategy compared to the number of cases without the controls. We observed that the control strategies resulted in a significant decrease in the number of affected population as against the increases in the case where there is no control. The control profile in figure 5(e) indicates that the optimal controls u_1, u_2, u_3, u_4 is at the upper bound when the use of face mask and physical distancing, sanitation, and treatment are at 100% implementation stage for 87 days, 98 days and 99 days respectively, and vaccination before it drops down to the lower bound

7.0 CONCLUSION

In this paper, we studied an SEIR structured model of covid-19 with vaccination and pets compartment and four (4) control strategies, being a modified version of the work of Chen *et al.* (2020). For the control parameters in the analytical solution, we presumed a constant control. The positivity of the solution was proven and the system of non-linear differential equations were epidemiologically and mathematically well-posed. Using the next generation matrix approach, we obtained the basic reproduction number. The disease free and endemic equilibrium were established. The disease free is locally asymptotically stable when $R_0 < 1$ and unstable if R_0 is greater than 1. The global stability of the disease free is proved to be globally asymptotically stable when the associated reproduction number $R_0 < 1$. This suggests that in a stable equilibrium the disease will entirely die out, however in an unstable equilibrium it will continue and spread to become endemic. In order to determine the ideal level needed to stop the spread of the covid-19 illness in a population, we employed Pontryagin's Maximum Principle. According to numerical findings, strategy V, which combines the use of vaccines, face masks and physical distancing, sanitation, and treatment, is

the most effective way for preventing the spread of the COVID-19 virus.

8.0 ACKNOWLEDGEMENT

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9.0 CONFLICT OF INTEREST

The authors declared no conflict of interest

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