



Mathematical Model on The Dynamics of Corruption Menace with Control Strategies

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ABSTRACT

In this study, we create a mathematical model of corruption dynamics that includes containment measures to stop the threat from spreading. We calculate the model's corruption-free equilibrium's basic reproduction number and stabilities. In order to maximize the number of incorruptible and honest persons while minimizing the dynamics of the threat, the model employed two control strategies: a public education campaign and the incarceration of those found guilty. Using Pontryagin's Maximum Principle (PMP), we determine the prerequisites for the best course of action for limiting the infection's spread. The numerical simulation that demonstrated the potential of control measures in preventing the dynamics of corruption in society from spreading was conducted using Runge-Kutta of order four.

Keywords: Dynamics, Menace, Enlightenment, Imprisonment, Honest

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1.0 INTRODUCTION

Corruption is an incident that slows down economics, social, political and human development of any society. A number of indices are available to quantify corruption. The most important is the Business International Index, which provides an estimate of the level of corruption in different nations and is compiled by the Economist Intelligence Unit. Other indexes include the Annual Political Risk Services Inc. (ICRG); the World Bank (WB) publishes the World Bank (WGI) and specifically its dimension on corruption (CCI); the Global Corruption Barometer (GCB), a public investigation; and the CPI, which is published by TI and gauges the extent of perceived corruption in the public sector. Furthermore, there are modern measurement experiments such as the Corruption Conviction Index and the Corruption Reflection Index (CRI) from the Institute for Corruption Studies (Dincer, 2020). Spyromitros and Panagiotidis (2022) examined two aspects of corruption: a) its assessment and b) its effects on the economic performance of 83 developing countries between 2012 and 2018 using AR (1) and FM-OLS data processing techniques. Moreover, the study's results show that corruption hinders the economic development of such developing countries. Varying levels of corruption have varying consequences on economic growth in different locations. For instance, corruption can have a positive or negative impact on growth in Latin American countries.

Damijan (2023) provides an overview of the concept of corruption throughout history, the models that are now being used to study it, and the main costs that corruption exacts on society at large and the economy. It evolved from a political problem to a public-private problem, then

from a public-political problem to a political-economic problem. Combating corruption started out as a moral struggle against a system of privilege, moved on to protecting crucial checks and balances, and is currently the main force behind state policy. In 2023, Zhang, Zhang, Zhang, Peng, Zhu, Liu, & You introduced a model that takes supervision levels and related expenses into account. The results show that when per capita income is low because of budgetary restrictions for anti-corruption control, the ideal degree of supervisory input, which maximizes government benefit, is almost zero in the case of homogeneous officials. However, the ideal degree of supervisory input abruptly increases to its maximum and then drops as per capita income rises above a particular threshold. Furthermore, when authorities have varying degrees of honesty, some circumstances lead to a middle-class per capita income range, which in turn produces an ideal amount of supervisory input and the phenomena known as "partial corruption. Thirty-two percent of Nigerian people who had at least one interaction with a public official in the 12 months before the 2019 study either gave or received a bribe from a public official. This indicates that while still extremely high, Nigeria's bribery prevalence has decreased somewhat since 2016, when it was recorded at 32.3%. Nevertheless, the decline is statistically significant. Since 2016, the prevalence of bribery has decreased in three of the six zones in the country: the North-West, which saw a significant (and statistically significant) decline from 36% to 25% of the total, while the other two zones saw smaller decreases, (United Nations Office on Drugs and Crime, 2019). High levels of corruption have a negative impact on productivity across all industries by lowering private investment and

causing uncertainty about the institutional environment. The empirical results demonstrate how capital flight might impair institutional capacity to manage corruption, degrade institutions with regard to environmental norms, and lower banking sector ratings by reducing productive investment, (United Nations Conference on Trade and Development, 2020). Therefore, several research works were carried-out to study the dynamic of corruption mathematically, its effects and the methods for control. We also review some research works on corruption that were carried out using Mathematical Model. A deterministic mathematical model explaining the dynamics of corruption's spread was created by Fantaye and Birhanu (2022), and it was examined by taking societal influence on morally upright people into account. The model's positivity and boundedness were established, and the next-generation matrix approach was used to get the basic reproduction number R_0 . The research demonstrates that for $R_0 < 1$, corruption-free equilibrium is asymptotically stable both locally and globally. Furthermore, whenever $R_0 > 1$, the endemic equilibrium point is asymptotically stable both locally and globally. Then, using MATLAB, various numerical simulations with and without optimal control is run to confirm the theoretical analysis. The model was then expanded to include optimal control. The best way to lessen the dynamic transmission of corruption is to use a prevention and punishment strategy, according to a numerical simulation of the optimum control model. Aloke (2023) looks at the dynamics of corruption and the three preventative measures put forth to deal with it in the Nigerian system. The Susceptible - Exposed - Corrupt - Jailed - Honest (SECJH) model used linear ODEs to characterize the dynamics of the corruption model. The reproduction number is used to determine the corruption eradication threshold. The Pontryagin's maximum concept was applied in the optimal control technique to evaluate the efficacy of suggested control measures. Through the use of a numerical technique called the Forward-Backward Sweep method and a MATLAB script designed for the method's implementation through the Runge-Kutta fourth order method, with the controls updated repeatedly for the changing values, the state and adjoint equations were numerically simulated. Their research demonstrates that implementing the suggested control measures can lead to a society free from corruption and that adhering to the requirements of the control measures is a successful anti-corruption strategy. Solomon, Emmanuel, Danat & Tsok (2023) examined the mechanisms underlying corruption and investigated preventative and remedial measures. It focuses specifically on reintegrating those who have recovered from corrupt practices back into society. Through an assessment of rehabilitation programs' efficacy, relapse

risk, and the impact of social support networks, the research aims to ascertain if the reintegration of formerly corrupt individuals results in a decrease in corruption or the resurgence of corrupt behaviors. The study looks into the spread of corruption among the populace using mathematical models and analytics. It investigates the stability characteristics of unchecked corruption models and the efficacy of various corruption prevention strategy combinations. By examining these variables, the study seeks to understand the fundamental dynamics of corruption and pinpoint tactics that can successfully lessen its incidence. A nonlinear deterministic corruption transmission dynamics with cost-effective techniques and optimal control analysis was proposed by Keno & Legesse (2023). They proved that the model solution is bounded and non-negative under a certain set of initial circumstances. Using the next generation matrix and the corruption-free equilibrium point, a fundamental reproductive number was determined. If the basic reproduction number is less than one, the corruption-free equilibrium point was asymptotically stable both locally and globally; if not, endemic corruption equilibrium appears. Additionally, the sensitivity of the model's parameters was examined, and the model showed forward bifurcation. Furthermore, by employing the Pontryagin minimum principle, the best interventions for minimising corruption are identified through the application of two control strategies: punishment and preventive. Finally, prevention is the most cost-effective and highly optimal approach to eliminating corruption, according to numerical prediction systems of optimality. Binuyo and Akinsola (2020) developed and examined is a mathematical model of the public's propagation of corruption dynamics. The local stability analysis of the mathematical model was presented along with the conditions for the stability of the corruption free equilibrium state. Alemneh (2020) created a nonlinear deterministic model for the dynamics of corruption, which is quantitatively examined using differential equation stability theory. The next generation matrix approach is used to obtain the basic reproduction number with respect to the corruption free equilibrium. The criteria for local and global asymptotic stability of corruption free and endemic equilibria were developed. Center manifold theory demonstrates forward bifurcation in the model. The model was expanded by rephrasing it as an optimum control problem and applying two (20 time defendant controls) to evaluate the effects of corruption on the general populace. These controls include media and commercial campaigns against corruption and the exposure of corrupt persons to punishment in jail. Pontryagin's Maximum Principle was developed in order to determine the prerequisites for the best possible control over the spread of corruption. Based on fractal-fractional notions and the Mittag-Leffler

kernel law, Awadallah, Rahman, Al-Duais, Al-Bossly, Abuasbe, and Arab (2023) provided a novel mathematical model for analyzing the persistence of corruption in human groups. Their model uses the corruption reproduction number to assess the stability of the corruption equilibrium points. In the context of a fractional fractal operator, they also used the fixed point theory idea to verify the existence and uniqueness of a solution. Lagrangian polynomials were used to produce an approximate solution, and the perturbed Ulam Hyers technique was used to confirm the stability of the solution. We simulate all compartments at various fractional orders and time durations to verify the correctness of our model and to offer further insights into the dynamics of corruption beyond natural orders. Akanni, Akinpelu, Olaniyi, Oladipo and Ogunsola (2019) created and examined a mathematical model for financial crime population dynamics that includes the best possible control strategies. It was determined what prerequisites financial crime steady states needed in order to exist and remain stable. To ascertain the relative significance of model parameters in the spread of financial crime, they performed sensitivity analysis. Moreover, two time-dependent optimal control strategies—a public education campaign (preventive) and a corrective measure—were evaluated using optimal control theory to see how they affected the dynamics of financial crime in a given population. Corruption is still a big issue in many nations throughout the world, despite efforts to minimize it in society, particularly in emerging nations. It has shown to be a significant barrier to economic development and progress in the nations with the greatest rates of corrupt activities. Any significant development will require more concerted efforts or strategies at the national and international levels to lower the incidence of corruption. In order to lessen the dynamics of corruption transmission, we offer several control measures and construct a mathematical model of the dynamics of corruption. Thus, we create a mathematical model of the dynamics of corruption that incorporates various control mechanisms, such as public education campaigns against corruption and the implementation of anti-corruption laws.

1.2 Formulation of the Model

The total population N at time t is composed of persons who are prone to corruption, forming sub-compartment $B(t)$; exposed people to corruption $E(t)$; people who refused to be corrupt $R(t)$; corrupt persons $C(t)$; people who are being investigated for allegedly corrupt activities $A(t)$; cleared of charges related to corruption $X(t)$; corrupt people who are found guilty and imprisoned $J(t)$; people who have overcome corruption $P(t)$ and

the morally upright people who abstained from corruption because they were honest.

It is assumed that corruptible individual's increases by the steady rate of recruiting is Λ , the people who were exonerated from corruption allegation after investigation at a rate ε and the proportion of persons who repented from corruption after serving the jail sentence and return to being a corrupt person. The susceptible individual population decreases by the force of infection (influence as a result of interaction with corrupt persons or with persons on corruption allegation) $\frac{(\rho C(t) + A(t))}{N(t)}$ (where

ρ is the modification perimeter that shows the strength of influence of the corrupt persons and $\rho > 1$) to become exposed to corruption. It is also decreased by the persons who die naturally at a rate μ .

The exposed compartment increases when people who were vulnerable came into contact with those who were corrupt or the persons who are under investigation due to the allegation of corruption on them $\frac{(\rho C(t) + A(t))}{N(t)}$,

where the modification perimeter $\rho > 1$. The decreases by the sum of the proportion of those who move to corrupt compartment and those who resisted to be corrupt at a rate δ .

It is believed that proportion θ of exposed persons who resists involving in any corrupt practices increases the population of corruption resistant compartment at a rate δ . The corruption resistant compartment decreases by those who became honest as a rate β and those that dies naturally at a rate μ .

The corrupt population increases by the proportion $(1-\theta)$ of persons who were exposed to corrupt practices and indulges in it at a rate δ . But it decreases who have corruption allegation at a rate σ and the naturally death rate μ .

The population of the persons under investigation as a result of corruption allegation increases at a rate σ . However, the population decreases by the proportion $\alpha\gamma$ of those who are found not guilty as a result of substantial evidence against them. While, μ represents the natural death rate in the population.

The number of people in the population who were exonerated increases by the proportion α of those that were investigated as a result of corruption allegation at a

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rate γ , however, it decreases at a rate determined by persons who return to the vulnerable compartment, ε and those that die naturally at a rate μ .

It is assumed that the population of jailed compartment increases by a proportion $(1-\alpha)$ of persons that were investigated as a result of corruption allegation and found to be guilty at a rate γ . While, ϕ represents the persons who served their jail term and moved to the recovered compartment and μ represents the natural death rate, μ . The repentant population increases due to movement of corruption ex-convict after serving their jail term at a rate ϕ . But it is decreases by the persons who either move to susceptible population or become honest after they repented at a rate κ and those that die naturally at a rate μ .

Honest individual population increases when some persons who desist from engaging in anything that will lead to the corruption at the rate β and the proportion of those who repented from corruption and become honest by developing resistant to corruption at a rate κ . The population decreases only by the natural death rate μ .

1.3 Model Equations

Therefore, the corruption model equations is given by

$$\left. \begin{array}{l} \frac{dB(t)}{dt} = \Lambda + \varepsilon X(t) + (1-\tau) \kappa R(t) - \frac{(\rho C(t) + A(t)) B(t)}{N(t)} - \mu B(t) \\ \frac{dE(t)}{dt} = \frac{(\rho C(t) + A(t)) B(t)}{N(t)} - (\delta + \mu) E(t) \\ \frac{dR(t)}{dt} = \theta \delta E(t) - (\beta + \mu) R(t) \\ \frac{dC(t)}{dt} = (1-\theta) \delta E(t) - (\sigma + \mu) C(t) \\ \frac{dA(t)}{dt} = \sigma C(t) - (\gamma + \mu) A(t) \\ \frac{dX(t)}{dt} = \alpha \gamma A(t) - (\varepsilon + \mu) X(t) \\ \frac{dJ(t)}{dt} = (1-\alpha) \gamma A(t) - (\phi + \mu) J(t) \\ \frac{dP(t)}{dt} = \phi J(t) - (\kappa + \mu) P \\ \frac{dH(t)}{dt} = \beta R(t) + \tau \kappa P(t) - \mu H(t) \end{array} \right\} \quad (1)$$

Table 1: Variables for the Model

Variables	Value and References
$B(t)$	79,640 Binuyo (2019)
$E(t)$	60,000 (Assumed)
$R(t)$	72,375 Binuyo (2019)
$C(t)$	17,000 (Assumed)
$A(t)$	39,970 EFCC (2022)
$X(t)$	2000 (Assumed)
$J(t)$	3785 EFCC (2022)

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$P(t)$	10,655 Binuyo (2019)
$H(t)$	1,500 Assumed

Tables 2: Parameters for the Model

Parameters	Value and References
Λ	0.001000 (Assumed)
ρ	1.20 (Assumed)
μ	0.01794 United Nations (2023)
θ	0.3 Lemecha (2018)
β	0.0004 Binuyo (2019)
δ	0.2 Alemneh (2020)
σ	0.008 (Assumed)
γ	0.0001 Binuyo (2019)
α	0.3 Alemneh (2020)
ε	0.00012 (Assumed)
ϕ	0.014 (Assumed)
κ	0.02 (Assumed)
τ	0.1 Lemecha (2018)

2.0 MODEL ANALYSIS

2.1 Basic Properties of the Model

We start with deriving the basic properties of the model

2.1.1 Positivity of the Solution

Theorem: Let the initial data be $\{B \geq 0, E \geq 0, R \geq 0, C \geq 0, A \geq 0, X \geq 0, J \geq 0, P \geq 0, H \geq 0\} \in \mathbb{R}_+^9$, then the solution set $\{B(t), E(t), R(t), C(t), A(t), X(t), J(t), P(t), H(t)\}$ is positive for all time $t > 0$.

Proof: Considering the first equation of the model which is;

$$\frac{dB}{dt} = \Lambda + \varepsilon X + (1-\tau) \kappa P - \frac{(\rho C + A)}{N} B - \mu B \quad (2)$$

$$\ln B \geq - \int \left(\frac{(\rho C + A)}{N} + \mu \right) dt + C \quad (3)$$

$$B(t) \geq B(0) e^{- \int \left(\frac{(\rho C + A)}{N} + \mu \right) dt} > 0$$

Similarly, considering the other equations in (1) by using the method above gives $E \geq 0, R \geq 0, C \geq 0, A \geq 0, X \geq 0, J \geq 0, P \geq 0, H \geq 0$ for all t .

2.1.2 Invariant Region

Theorem: The solutions to the system (1) with initial conditions and the biological feasible region

$$\Omega^* = \left\{ (B, E, R, C, A, X, J, P, H) \in \mathbb{R}_+^9; N \leq \frac{\Lambda}{\mu} \right\}$$

positively invariant and attractive with respect to the system (1).

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Proof: Summing the model equations in (1) gives

$$\frac{dN}{dt} = \Lambda - \mu N$$

where, $N = B + E + R + C + A + X + J + P + H$

since $\frac{dN}{dt} = \Lambda - \mu N$, it follows that $\frac{dN}{dt} \leq 0$ if $N \geq \frac{\Lambda}{\mu}$.

By employing the standard comparison theorem by Lakshmikantham, V., Leela, S. & Martynyuk, A. A. (2000) be utilized to demonstrate that;

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

separating the variables

$$\frac{dN}{dt} + \mu N = \Lambda$$

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu} \left[1 - e^{-\mu t} \right]$$

Particularly, $N(t) \leq \frac{\Lambda}{\mu}$ if $N(0) \leq \frac{\Lambda}{\mu}$, this implies that the

region Ω^* is favorably invariant.

Consequently, the model in the domain Ω^* , is mathematically well-posed, as the variables that are employed in the model as a whole (1), proves to be positive for $t > 0$. It is therefore, adequate to take into account the dynamics of the flow produced by the model (1) in the domain Ω^* .

2.2 Equilibrium Point

We set (1) to be equal to zero;
at corruption free;

$$M_E = (B^*, E^*, R^*, C^*, A^*, X^*, J^*, P^*, H^*) \\ = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad (4)$$

2.3 Reproduction Number

This is the anticipated number of secondary cases that a typical infectious individual would cause in a community that is fully susceptible. (van den Driessche and Watmough, 2002).

We compute the inverse of (6) to obtain;

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Theorem: It is claimed that the disease free equilibrium (DFE) point is locally asymptotically steady if $R_0 < 1$ as well as unsteady if $R_0 > 1$.

Proof: In order to demonstrate the theorem, we acquire and assess the non-negative matrix F of the non-singular matrix and new words related to infections V of transition terms for (1) at DFE as follows;

$$R_0 = \rho(FV^{-1}) \text{ where } \rho(FV^{-1}) \text{ denotes the spectral radius with respect to the matrix } FV^{-1},$$

$F_i(x)$ be the frequency of new infections occurring in the compartment i and $V_i(x)$ be the frequency of people moving out of the compartment i

$$F = \begin{pmatrix} (\rho C + A)B \\ N \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } -V = \begin{pmatrix} (\delta + \mu)E \\ -(1-\theta)\delta E + (\sigma + \mu)C \\ -\sigma C + (\gamma + \mu)A \\ -(1-\alpha)\gamma A + (\phi + \mu)J \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & \rho \frac{B^*}{N^*} & \frac{B^*}{N^*} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

and

$$V = \begin{pmatrix} (\delta + \mu) & 0 & 0 & 0 \\ -(1-\theta)\delta & (\sigma + \mu) & 0 & 0 \\ 0 & -\sigma & (\gamma + \mu) & 0 \\ 0 & 0 & -(1-\alpha)\gamma & (\phi + \mu) \end{pmatrix} \quad (6)$$

Evaluating (5) at corruption free equilibrium (CFE) gives;

$$F = \begin{pmatrix} 0 & \rho & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

$$V^{-1} = \begin{pmatrix} \frac{1}{\delta + \mu} & 0 & 0 & 0 \\ \frac{\delta - \theta\delta}{(\delta + \mu)(\sigma + \mu)} & \frac{1}{\sigma + \mu} & 0 & 0 \\ \frac{\sigma(\delta - \theta\delta)}{(\sigma + \mu)(\delta + \mu)(\gamma + \mu)} & \frac{\sigma}{(\sigma + \mu)(\gamma + \mu)} & \frac{1}{\gamma + \mu} & 0 \\ \frac{\sigma(\delta - \theta\delta)(\gamma - \alpha\gamma)}{(\gamma + \mu)(\sigma + \mu)(\delta + \mu)(\phi + \mu)} & \frac{\sigma(\gamma - \alpha\gamma)}{(\gamma + \mu)(\sigma + \mu)(\phi + \mu)} & \frac{\gamma - \alpha\gamma}{(\gamma + \mu)(\phi + \mu)} & \frac{1}{\phi + \mu} \end{pmatrix} \quad (8)$$

We multiply (6) with (7) to have;

$$FV^{-1} = \begin{pmatrix} \frac{\rho(\delta - \theta\delta)}{(\delta + \mu)(\sigma + \mu)} + \frac{\sigma(\delta - \theta\delta)}{(\sigma + \mu)(\delta + \mu)(\gamma + \mu)} & \frac{\rho}{\sigma + \mu} + \frac{\sigma}{(\sigma + \mu)(\gamma + \mu)} & \frac{1}{\gamma + \mu} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

From (9) we obtain;

$$R_0 = \sqrt{\frac{\delta\gamma\rho + \delta\mu\rho - \gamma\rho\theta\delta - \mu\rho\theta\delta + \delta\sigma - \sigma\theta\delta}{\delta\gamma\mu + \delta\gamma\sigma + \sigma\mu^2 + \delta\mu\sigma + \gamma\mu^2 + \gamma\mu\sigma + \mu^2\sigma + \mu^3}}$$

2.4 Global Stability of CFE

using the equation above (1);

we use the method adopted by Alhassan, *et. al.*, (2021), considering the given conditions below;

$$\frac{dy}{dt} = F(Y, Z) \quad (10)$$

$$\frac{dz}{dt} = G(Y, Z); \quad G(Y, 0) = 0 \quad (11)$$

where $Y = (B, R, X, P, H)$ and $Z = (E, C, A, J)$; $Y \in R^5$ denotes the incorrupt persons while $Z \in R^4$ denotes the corrupts population

$$M_E = \begin{pmatrix} \frac{\Lambda}{\mu}, 0, 0, 0, 0, 0, 0, 0, 0 \end{pmatrix} \quad (12)$$

The two conditions to be met for the global stability are;

(i) $\frac{dY}{dt} = F(Y, 0)$, Y^* is globally asymptotically stable (G.A.S)

(ii) $\frac{dZ}{dt} = AZ - G(Y, Z) \geq 0$ for $(X, Z) \in \Pi$

$A = D_2 G(Y, 0)$ is an M - matrix (the off diagonal elements of A are non-negative) and Π is the feasible region.

Condition I

$$\frac{dY}{dt} = F(Y, 0) = \begin{pmatrix} \Lambda - \mu B \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (13)$$

$$\frac{dB}{dt} + \mu B = \Lambda \quad (14)$$

$$B(t) = B(0)e^{-\mu t} + \frac{\Lambda}{\mu} [1 - e^{-\mu t}] \quad (15)$$

From (15), $B(t) \rightarrow N^*(t)$ as $t \rightarrow \infty$ whatever the value of $B(0)$.

Therefore, $Y^* = \begin{bmatrix} \frac{\Lambda}{\mu} \\ B \end{bmatrix}$ is globally asymptotically stable

Condition II

$$G(Y, Z) = AZ - G(Y, Z), \quad G(Y, Z) \geq 0 \text{ for } (Y, Z) \in \Omega$$

$$AZ = \begin{pmatrix} -(\delta + \mu) & \rho & 1 & 0 \\ (1-\theta)\delta & -(\sigma + \mu) & 0 & 0 \\ 0 & \sigma & -(\gamma + \mu) & 0 \\ 0 & 0 & (1-\alpha)\gamma & -(\phi + \mu) \end{pmatrix} \quad (16)$$

$$G(Y, Z) = \begin{pmatrix} (\rho C + A) \frac{B}{N} - (\delta + \mu) E \\ (1-\theta)\delta E - (\sigma + \mu) C \\ \sigma C - (\gamma + \mu) A \\ (1-\alpha)\gamma A - (\phi + \mu) J \end{pmatrix} \quad (17)$$

$$G(Y, Z) = AZ - G(Y, Z)$$

$$G(Y, Z) = \begin{pmatrix} (\rho C + A) \left(1 - \frac{B}{N}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $0 \leq B \leq N$, it shows that $\hat{G}(Y, Z) \geq 0$. This also shows clearly that $Y^* = \left(\frac{\Lambda}{\mu}, 0\right)$ is globally asymptotically stable equilibrium of $\frac{dY}{dt} = F(Y, 0)$. Therefore, M_E is globally asymptotically stable.

3.0 Optimal Control

By introducing time-dependent control variables $u_1(t)$ and $u_2(t)$ in (1), the optimal control problem is formulated to ascertain the most effective techniques (Public enlightenment and Imprisonment) for controlling corruption with minimal implementation implication. We seek optimal control u_1, u_2 such that

$$J(u_1^*, u_2^*) = \min \{J(u_i, u_2), (u_i, u_2 \in U)\} \quad (18)$$

The requirements that must be met for an optimal control in order for it to comply with the Pontryagin's Maximum Principle (PMP), (Lenhart, & Workman, 2007) that converts (20) into a pointwise minimization problem for Hamiltonian H_T with respect to u_1, u_2 . The objective functional is given as

$$J = \min_{u_1, u_2} \int_0^{tf} \left(F_1 C + F_2 A + \frac{G_1 u_1^2}{2} + \frac{G_2 u_2^2}{2} \right) dt \quad (19)$$

where tf is the final time and the coefficients F_1, F_2, G_1 and G_2 are balanced by positive weights. The terms $F_1 C$ and $F_2 A$ are the expense of corruption while $G_1 u_1^2$ and $G_2 u_2^2$ are the costs associated with $u_1(t)$ (public enlightenment) and u_2 (imprisonment).

Subject to;

$$\left. \begin{aligned} \frac{dB(t)}{dt} &= \Lambda + \varepsilon X(t) + (1-\tau) \kappa R(t) - (1-u_1) \frac{(\rho C(t) + A(t)) B(t)}{N(t)} - \mu B(t) \\ \frac{dE(t)}{dt} &= (1-u_1) \frac{(\rho C(t) + A(t)) B(t)}{N(t)} - (\delta + \mu) E(t) \\ \frac{dR(t)}{dt} &= \theta \delta E(t) - (\beta + \mu) R(t) \\ \frac{dC(t)}{dt} &= (1-\theta) \delta E(t) - (u_2 \sigma + \mu) C(t) \\ \frac{dA(t)}{dt} &= u_2 \sigma C(t) - (\gamma + \mu) A(t) \\ \frac{dX(t)}{dt} &= \alpha \gamma A(t) - (\varepsilon + \mu) X(t) \\ \frac{dJ(t)}{dt} &= (1-\alpha) \gamma A(t) - (\phi + \mu) J(t) \\ \frac{dP(t)}{dt} &= \phi J(t) - (\kappa + \mu) P(t) \\ \frac{dH(t)}{dt} &= \beta R(t) + \tau \kappa P(t) - \mu H(t) \end{aligned} \right\} \quad (20)$$

We take the control variables $u(t) = [u_1(t), u_2(t)] \in U$ into consideration given the optimal control system problem in relation to the state variables $B(t), E(t), R(t), C(t), A(t), X(t), J(t), P(t), H(t)$ where the bounded and measured control variables are $\{U = (u_1, u_2)\}$ is Lebesgue measurable on $[0, 1]$, $0 \leq u_i(t) \leq 1$, $t \in [0, tf]$, $i = 1, 2$

Theorem 4.1 Lenhart and Workman (2007)

If $u_i^*(t)$ and $x^*(t)$ are optimal for problem, then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that $H_T(t, x^*(t), u(t), \lambda(t)) \leq H_T(t, x^*(t), u^*(t), \lambda(t))$ For all controls u at each time t , where the Hamiltonian H_T is

$$H = f(t, x(t), u(t) + \lambda(t) g(t, x(t), u(t))),$$

and

$$\lambda'(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x},$$

$$\lambda(t) = 0.$$

Thus, our Hamiltonian is

$$\left. \begin{aligned} H_T &= F_1 C + F_2 A + \frac{G_1 u_1^2}{2} + \frac{G_2 u_2^2}{2} \\ &+ \lambda_B \left[\Lambda + \varepsilon X + (1-\tau) \kappa P - (1-u_1) \frac{(\rho C + A) B}{N} - \mu B \right] \\ &+ \lambda_E \left[(1-u_1) \frac{(\rho C + A) B}{N} - (\delta + \mu) E \right] \\ &+ \lambda_R \left[\delta \theta E - (\beta + \mu) R \right] \\ &+ \lambda_C \left[(1-\theta) \delta E - (u_2 \sigma + \mu) C \right] \\ &+ \lambda_A \left[u_2 \sigma C - (\gamma + \mu) A \right] \\ &+ \lambda_X \left[\alpha \gamma A - (\varepsilon + \mu) X \right] \\ &+ \lambda_J \left[(1-\alpha) \gamma A - (\phi + \mu) J \right] \\ &+ \lambda_P \left[\phi J - (\kappa + \mu) P \right] \\ &+ \lambda_H \left[\beta R + \tau \kappa P - \mu H \right] \end{aligned} \right\} \quad (21)$$

where λ_B , λ_E , λ_R , λ_C , λ_A , λ_X , λ_J , λ_P and λ_H are also referred to as co-state or adjoint variables.

We use the Pontryagin's Maximum Principle and the existence result for the optimal control as stated by (Lenhart & Workman, 2007) to determine the prerequisites for this optimal control.

Theorem 4.2

The optimal controls u_1^* , u_2^* and solutions B , E , R , C , A , X , J , P , H of the corresponding state system (21) that minimizes $J(u_1, u_2)$ over Π . Then there exists adjoint variables $\lambda_B, \lambda_E, \lambda_R, \lambda_C, \lambda_A, \lambda_X, \lambda_J, \lambda_P$ and λ_H satisfying

$$\left. \begin{aligned} \frac{d\lambda_B}{dt} &= -\left[-(1-u_1) \frac{(\rho C + A)}{N} \lambda_B - \mu \lambda_B + (1-u_1) \frac{(\rho C + A)}{N} \lambda_E \right] \\ \frac{d\lambda_E}{dt} &= -\left[-(\delta + \mu) \lambda_E + \delta \theta \lambda_R + (1-\theta) \delta \lambda_C \right] \\ \frac{d\lambda_R}{dt} &= -\left[(\beta + \mu) \lambda_R + \beta \lambda_H \right] \\ \frac{d\lambda_C}{dt} &= -\left[F_1 - (1-u_1) \frac{\rho B}{N} \lambda_B + (1-u_1) \frac{\rho B}{N} \lambda_E - (u_1 \sigma + \mu) \lambda_C + u_2 \sigma \lambda_A \right] \\ \frac{d\lambda_A}{dt} &= -\left[F_2 - (1-u_1) \frac{B}{N} \lambda_B + (1-u_1) \frac{B}{N} \lambda_E - (\gamma + \mu) \lambda_A + \alpha \gamma \lambda_X + (1-\alpha) \lambda_J \right] \\ \frac{d\lambda_X}{dt} &= -\left[\varepsilon \lambda_B - (\varepsilon + \mu) \lambda_X \right] \\ \frac{d\lambda_J}{dt} &= -\left[-(\phi + \mu) \lambda_J + \phi \lambda_P \right] \\ \frac{d\lambda_P}{dt} &= -\left[(1-\tau) \kappa \lambda_B - (\kappa + \mu) \lambda_P + \tau \kappa \lambda_H \right] \\ \frac{d\lambda_H}{dt} &= -\left[-\mu \lambda_H \right] \end{aligned} \right\} \quad (22)$$

with transversality conditions:

$$\begin{aligned} \lambda_B(t_f) &= \lambda_E(t_f) = \lambda_R(t_f) = \lambda_C(t_f) = \lambda_A(t_f) \\ &= \lambda_X(t_f) = \lambda_J(t_f) = \lambda_P(t_f) = \lambda_H(t_f) = 0 \end{aligned} \quad (23)$$

And the controls u_1^* and u_2^* satisfy the optimality conditions:

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left(1, \frac{\partial H_T}{\partial u_1} \right) \right\} \\ u_2^* &= \max \left\{ 0, \min \left(1, \frac{\partial H_T}{\partial u_2} \right) \right\} \end{aligned} \quad (24)$$

Proof

By differentiating the Hamiltonian function, which is evaluated at the optimal control, the differentiable equations controlling the adjoint variables can be found. The adjoint system can therefore be expressed as;

$$\begin{aligned} -\frac{d\lambda_B}{dt} &= \frac{\partial H}{\partial B(t)}, \quad -\frac{d\lambda_E}{dt} = \frac{\partial H_T}{\partial E(t)}, \quad -\frac{d\lambda_R}{dt} = \frac{\partial H_T}{\partial R(t)}, \\ -\frac{d\lambda_C}{dt} &= \frac{\partial H_T}{\partial C(t)}, \quad -\frac{d\lambda_A}{dt} = \frac{\partial H_T}{\partial A(t)}, \quad -\frac{d\lambda_X}{dt} = \frac{\partial H_T}{\partial X(t)}, \\ -\frac{d\lambda_J}{dt} &= \frac{\partial H_T}{\partial J(t)}, \quad -\frac{d\lambda_P}{dt} = \frac{\partial H_T}{\partial P(t)} \quad \text{and} \quad -\frac{d\lambda_H}{dt} = \frac{\partial H_T}{\partial H(t)} \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial H_T}{\partial B} &= (1-u_1) \frac{(\rho C + A)}{N} (\lambda_B - \lambda_E) + \mu \lambda_B \\ \frac{\partial H_T}{\partial E} &= \delta (\lambda_E - \lambda_C) + \delta \theta (\lambda_C - \lambda_E) + \mu \lambda_E \\ \frac{\partial H_T}{\partial R} &= \beta (\lambda_R - \lambda_H) + \mu \lambda_R \\ \frac{\partial H_T}{\partial C} &= (1-u_1) \frac{\rho B}{N} (\lambda_B - \lambda_E) + u_2 \sigma (\lambda_C - \lambda_A) + \mu \lambda_C - F_1 \\ \frac{\partial H_T}{\partial A} &= (1-u_1) \frac{B}{N} (\lambda_B - \lambda_E) + \gamma (\lambda_A - \lambda_J) + \alpha \gamma (\lambda_J - \lambda_X) + \mu \lambda_A - F_2 \\ \frac{\partial H_T}{\partial X} &= \varepsilon (\lambda_X - \lambda_B) + \mu \lambda_X \\ \frac{\partial H_T}{\partial J} &= \phi (\lambda_J - \lambda_P) + \mu \lambda_J \\ \frac{\partial H_T}{\partial P} &= \kappa (\lambda_P - \lambda_B) + \tau \kappa (\lambda_B - \lambda_H) + \mu \lambda_P \\ \frac{\partial H_T}{\partial H_f} &= \mu \lambda_H \end{aligned} \right\} \quad (25)$$

then, with transversality conditions:

$$\begin{aligned} \lambda_B(t_f) &= \lambda_E(t_f) = \lambda_R(t_f) = \lambda_C(t_f) = \lambda_A(t_f) \\ &= \lambda_X(t_f) = \lambda_J(t_f) = \lambda_P(t_f) = \lambda_H(t_f) = 0 \end{aligned} \quad (26)$$

Hence, solving $\frac{\partial H_T}{\partial u_1} = 0$, $\frac{\partial H_T}{\partial u_2} = 0$, gives the characterization of controls.

$$\frac{\partial H_T}{\partial u_1} = -G_1 u_1^* + \frac{(\rho C^* + A^*) B^*}{N} (\lambda_E - \lambda_B)$$

$$u_1^* = \frac{(\rho C^* + A^*) B^*}{G_1 N} (\lambda_E - \lambda_B)$$

$$\frac{\partial H_T}{\partial u_2} = -G_2 u_2^* + \sigma C^* (\lambda_C - \lambda_A)$$

$$u_2^* = \frac{\sigma C^* (\lambda_C - \lambda_A)}{G_2}$$

Based on the standard controls argument, we conclude that the bound on the controls are;

$$\begin{aligned} u_1^* &\begin{cases} 0 & \text{if } u_1^* \leq 0 \\ u_1^c & \text{if } 0 < u_1^* < 1 \\ 1 & \text{if } u_1^* \geq 1 \end{cases} & u_2^* &\begin{cases} 0 & \text{if } u_2^* \leq 0 \\ u_2^c & \text{if } 0 < u_2^* < 1 \\ 1 & \text{if } u_2^* \geq 1 \end{cases} \end{aligned}$$

The uniqueness of the optimality system (23) – (25) is attained because the solutions to the state and adjoint equations are apriori bounded. Therefore, the duration of

the break is limited to $[0, t_f]$ in order to ensure the optimality system's uniqueness. As demonstrated by Kirschner, Lenhart, and Serbin (1997) and Joshi (2002), the constraint is typical in control problems.

4.0. NUMERICAL SIMULATIONS

This section discusses how interventions affect a population's ability to control the spread of corruption. The forward-backward iterative method was used to solve the optimality system, leading to the best public education and incarceration policies for preventing the development of corruption in society. After the state system and the adjoint system in the optimality system were solved, the optimal control solution was found. We employ the fourth-order Runge–Kutta approach to solve the state equations with a presumption for the controls over the simulated time because of the transversality conditions. To observe the impact of each control method in their respective compartments, we implement public education against corruption as the first control strategy for the vulnerable class and the imprisonment of the corrupt individuals upon being proven guilty of the corruption.

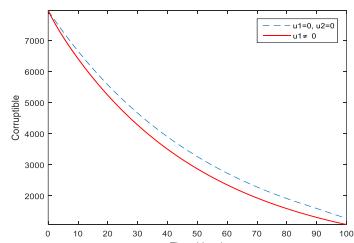


Figure 5.1(a)

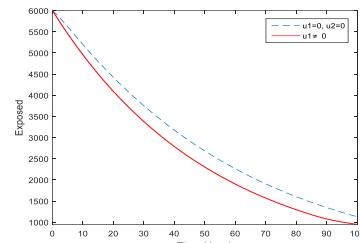


Figure 5.1(b)

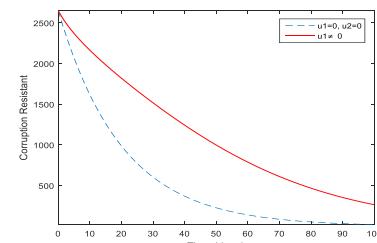


Figure 5.1(c)

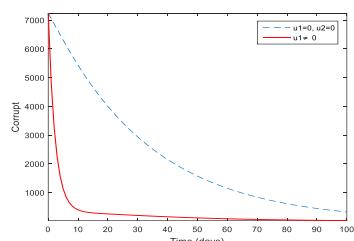


Figure 5.1(d)

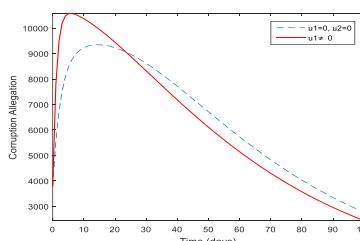


Figure 5.1(e)

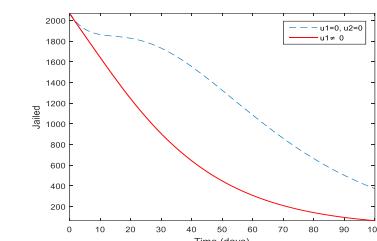


Figure 5.1(f)

4.1 Executing u_1 as Control Strategy

As a control measure against the rise of corruption in society, we carry out public education campaigns against it. We apply u_1 to reduce corruption by maximizing the functional (J) aim while setting $u_2 = 0$ which shows the impact of the control in curtailing the corruption transmission in the society. We notice from figure 5.1(a) and figure 5.1(b) some considerable decrease of the corruptible and exposed persons due to implementation of the control. While, figure 5.1(c) demonstrates a notable rise in the number of people who oppose corruption when control measures are used as opposed to when none are used. Figure (d) shows rapid decrease of corrupts persons due to implementation of the public enlightenment u_1 as a control strategy. However, figure 5.1(a), 5.1(b), 5.1(d), 5.1(e) and 5.1(f) indicate the decrease in corruption due to u_1 which demonstrates how well the control is working to stop corruption from spreading among the populace. Figures 5.1(c) and 5.1(g) shows some significant increase in both corruption resistant and honest persons as a result of implementing public enlightenment as a control strategy, while figure 5.1(h) the control profile demonstrates a sharp decrease in corruption from 100% to the lower bound at $t = 95$ days.

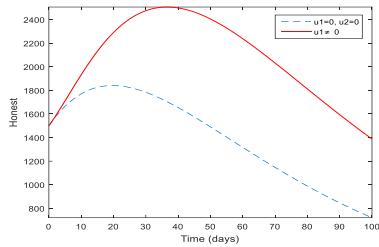


Figure 5.1(g)

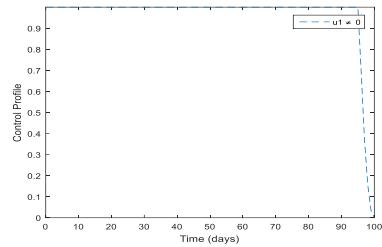


Figure 5.1(h)

4.2 Implementing (u_2) as Control Strategy

When determining the objective functional (J), the imprisonment of those found guilty of corruption was used to maximize when $u_1 = 0$. It has been observed from figures 5.2(a), 5.2(b), 5.2(d) and 5.2(e) that the number of Corruptible, exposed, allegation, jailed persons go down swiftly due to the effect of the control as against to the situation where there is no control. Figures 5.2(c), 5.2(f) and 5.2(g) describe the effectiveness of the control in reducing corruption in the society. While, Figure 5.2(h) of the control profile illustrates how effective the control over the incarceration of individuals guilty of corruption is in reducing the spread from the upper bound to the lower bound at $t = 97$ days.

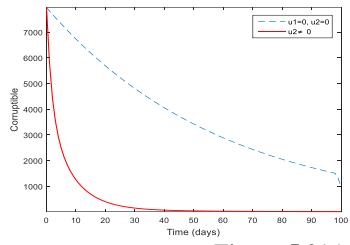


Figure 5.2(a)

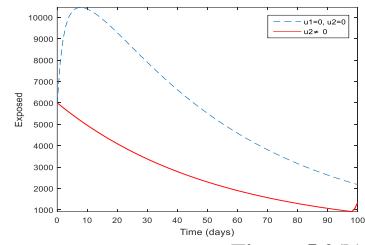


Figure 5.2(b)

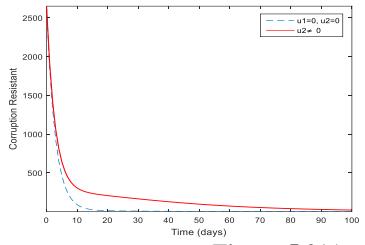


Figure 5.2(c)

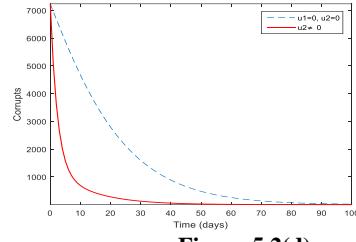


Figure 5.2(d)

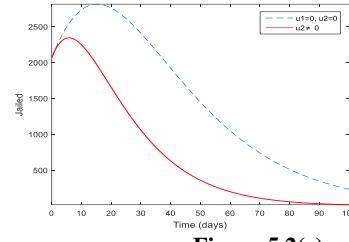


Figure 5.2(e)

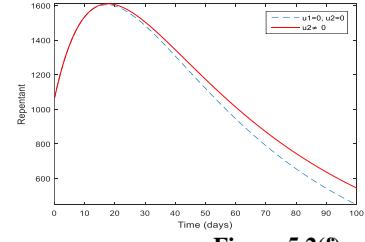


Figure 5.2(f)

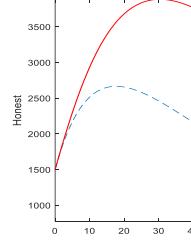


Figure 5.2(g)

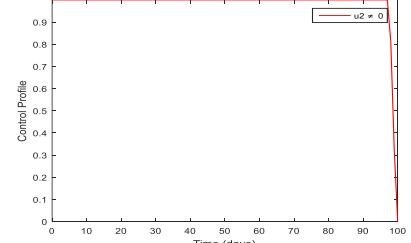


Figure 5.2(h)

4.3 Implementing u_1 and u_2 as control strategy

The controls u_1 and u_2 are used to simulate how well the two controls work simultaneously by optimizing the objective functional (J). We saw in figures 5.3(a) and 5.3(b). 5.3(d), 5.3(e), and 5.3(f) illustrate the impact of implementing the two (2) controls simultaneously and demonstrate how the gradual declines in corruption balance against the lack of control. While figures 5.3(c), 5.3(g) and 5.3(h) signify increase in number of corruption resistant, honest and repentant persons due to the use of the controls simultaneously. On the other hand, the control profile (figure 5.3(i)) indicates a rapid decline in corruption from the upper bound to the lower bound after about 100 days.

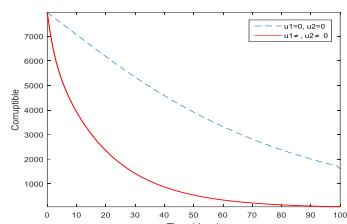


Figure 5.3(a)

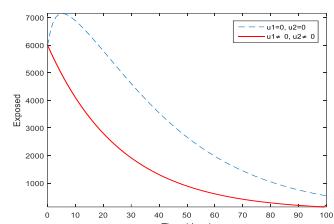


Figure 5.3(b)

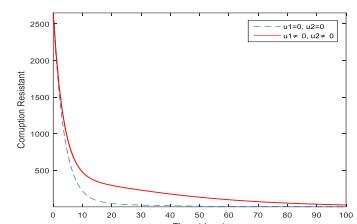


Figure 5.3(c)

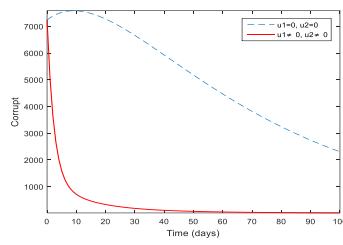


Figure 5.3(d)

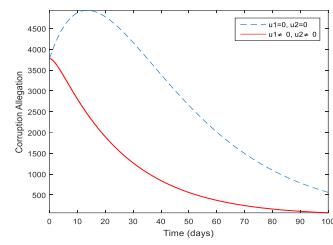


Figure 5.3(e)

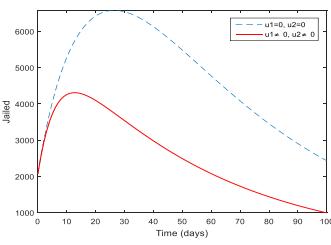


Figure 5.3(f)

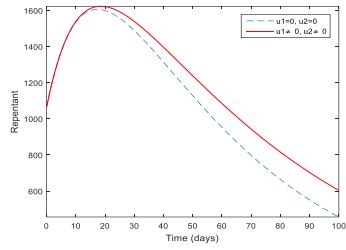


Figure 5.3(g)

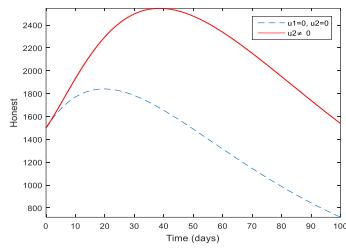


Figure 5.3(h)

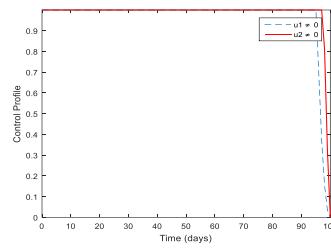


Figure 5.3(i)

5.0 CONCLUSION

To investigate the impacts of corruption, a mathematical model describing its dynamics was developed. The computation of corruption equilibrium was applied to both the local and global stabilities of the CFE. Two time-dependent control factors were also incorporated to provide an ideal control that will help stop the corruption from spreading throughout the population. A public education campaign against corruption and the punishment of those found guilty of corruption were the two (2) controls that we examined for effectiveness using Pontryagin's maximal principle. Following a certain amount of time, it has been noted that implementing the control(s) either separately or simultaneously can significantly slow the dynamics of corruption within the community.

REFERENCES

Akanni, J. O., Akinpelu, F. O., Olaniyi, S., Oladipo, A. T. and Ogunsola, A. W. (2019) Modelling Financial Crime Population Dynamics: Optimal Control and Cost-Effectiveness Analysis. International Journal of Dynamics and Control, Springer-Verlag GmbH Germany. <https://doi.org/10.1007/s40435-019-00572-3>

Alemneh, H. T. (2020) Mathematical Modeling, Analysis, and Optimal Control of Corruption Dynamics, *J. Appl. Math.* 2020, 5109841. <https://doi.org/10.1155/2020/5109841>.

Alhassan, A., Momoh, A. A., Abdullahi, S. A. & Audu, A. (2021) Mathematical Model for the Transmission Dynamics of Typhoid Fever with Treatment. *International Journal of Science for Global Sustainability*, Vol. 7(2), pp. 37–49

Aloke, S. N (2023) Analyzing Corruption Dynamics and Control Measures in Nigeria: A Mathematical Model Approach. *Asian Journal of Pure and Applied Mathematics*, Volume 5(1), Page 493-511; Article no. AJPAM.1422

Awadallah, M., Rahman, M., Al-Duais, F. S., Al-Bossly, A., Abuasbe, K. & Arab M (2023) Exploring the role of fractal-fractional operators in mathematical modelling of corruption. *Applied Mathematics in Science and Engineering*, Volume 31(1).

Binuyo, A. O. (2019) Eigenvalue Elasticity and Sensitivity Analyses of the Transmission Dynamic

IJSGS FUGUSAU VOL. 10 (1)

Model of Corruption. *Journal of the Nigerian Society of Physical Sciences*, 1 (2019) 30–34.

Binuyo, A. Y. & Akinsola, V. O. (2020) Stability Analysis of the Corruption Free Equilibrium of the Mathematical Model of Corruption in Nigeria. *Mathematical Journal of Interdisciplinary Sciences*, 8(2), 61–69.

Castillo-Chavez, C., Feng, Z. & Huang, W., (2002) On the computation of Ro and its Role on Global Stability, in: Castillo-Chavez C., Blower S., Van den Driessche P., Krirschner D. & Yakubu A.A.(Eds), *Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction*. The IMA Volumes in Mathematics and its Applications. Springer-Verlag, New York, 125, 229-250.

Dineer, O. (2020). If you're corrupt, you'd better be free. *Journal of Economic Studies*, 47(6), 1307–1325. <https://doi.org/10.1108/JES-04-2019-0153>

Economic and Financial Crime Commission (2022) Convictions. Retrieved on 8th February, 2023 from https://www.efcc.gov.ng/efcc/images/pdfs/3785_Convictions_recorded_in_2022.pdf.

Fantaye, A. K. & Zerihun Kinfe Birhanu, Z. K. (2022) Mathematical Model and Analysis of Corruption Dynamics with Optimal Control. *Journal of Applied Mathematics*, Volume 2022, Article ID 8073877. DOI: <https://doi.org/10.1155/2022/8073877>.

Joshi, H. R. (2002). Optimal Control of an HIV Immunology Model, *Optim. Control Appl. Math.*, 23, 199–213.

Keno, T. D. & Legesse, F. M. (2023) Modelling and Optimal Control Strategies of Corruption Dynamics. *Appl. Math. Inf. Sci.* 17, No. 4, 673-683 .

Kirschner, D., Lenhart, S. & Serbin, S. (1997). Optimal Control of the Chemotherapy of HIV. *Journal of Math. Biol.* 35, 775–792.

Lemecha, L (2018) Modelling corruption dynamics and its analysis,” *Ethiopian Journal of Sciences and Sustainable Development*, vol. 5, no. 2, pp. 13–27.

WEBSITE: <https://fugus-ijsgs.com.ng>

Lenhart, S. & Workman, J. T. (2007). *Optimal Control Applied to Biological Models*, Mathematical and Computational Biology Series, Chapman & Hall/CRC Press, London/Boca Raton

Solomon, I. R., Emmanuel, S., Danat N. T., & Tsok S. H. (2023). Mathematical Modeling of Corruption Dynamics: Examining the Reintegration of Formerly Corrupt Individuals. *FUDMA Journal of Sciences*, 7(4), 1 - 13. <https://doi.org/10.33003/fjs-2023-0704-1888>.

Spyromitros, E. & Panagiotidis, M. (2022) The impact of corruption on economic growth in developing countries and a comparative analysis of corruption measurement indicators, *Cogent Economics & Finance*, 10:1, DOI: 10.1080/23322039.2022.2129368 .

United Nations Conference on Trade and Development (2020). *Economic Development in Africa Report 2020: Tackling Illicit Financial Flows for Sustainable Development in Africa*, UN.

United Nations Office on Drugs and Crime (2019) *Corruption in Nigeria: Patterns and Trends*. Second survey on corruption as experienced by the population.

United Nations (2023) *Nigeria Death Rate 1950-2023* Retrieved on 2023-02-05 from <https://www.macrotrends.net/countries/NGA/nigeria/death-rate>

van den Driessche, P & Watmough, J. (2002) Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences* 180 (2002) 29–48.

Zhang, M., Zhang, H., Zhang, L., Peng, X., Zhu, J., Liu, D. & You, S. (2023) Corruption, anti-corruption, and economic development. *Humanities and Social Sciences Communications*, volume 10, Article number: 434. DOI: <https://doi.org/10.1057/s41599-023-01930-5>