

Classical And Bayesian Inferences of a New Additive Weibull Extension Distribution with Application

¹A. S. Mohammed, ²N. Silas, ³I. Abdullahi, ⁴J. Abdullahi and ^{5*}B. Abba

^{1,4}Department of Statistics, Ahmadu Bello University, Zaria-Nigeria

²Department of Mathematics, Institute of Applied Pedagogy, University of Burundi, Avenue MWEZI GISABO, No \$, P.O. Box 5223, Bujumbura-Burundi.

^{3,5}Department of Mathematics, Yusuf Maitama Sule University, Kano

⁵School of Mathematics and Statistics, Central South University, Hunan, China.

*Corresponding Author's Email: badamasiabba@csu.edu.cn

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ABSTRACT

For the past few decades, many generalizations of Weibull distributions have been defined. The generalizations (modifications or extensions) are necessary to allow the resulting distributions provide a better fit for both monotones (increasing and decreasing) and non-monotone (formed like a bathtub-shaped and up-side-down bathtub) failure rate (FR) lifetime data sets available in the literature, which the classical Weibull distribution cannot model appropriately. In this article, another approach is adopted to extend the well-known additive Weibull (AddW) distribution to introduce a new five-parameter model named as generalized extended additive Weibull (GExAddW) distribution. The new model is constructed by re-defining the AddW model to a three-parameter model based on two alternate parametric forms of the Weibull reliability functions and then added two extra positive shape parameters to the resulting model to form the GExAddW distribution. Both classical and Bayesian inferences were carried out for the derived model. The GExAddW could be a viable candidate for modeling complex failure times. Moreover, with intricate nature of the model, we suggest using numerical approaches for its property's computation. A detailed account of the distribution properties was presented. When modeling lifetime data with a non-monotonic failure rate shape, the GExAddW distribution fit the data better than alternative lifetime distributions.

Keywords: Generalization, failure rate, Weibull distribution, lifetime, Bayesian inferences.

1.0 INTRODUCTION

Hundredth of the Weibull model's generalizations as lifetime distributions have been defined for about three decades to the present. The generalizations (modifications or extensions) are necessary to allow the resulting distributions provides a better fit for both monotones (increasing and decreasing) and non-monotone (bathtub-shaped and up-side-down bathtub) failure rate (FR) lifetime data sets available in the literature, which the classical Weibull distribution cannot model appropriately. Bathtub-shaped FR function is considered the most important in reliability engineering as many failure time data sets have bathtub FR shapes. Thus, distributions with this type of FR used in this regard. One primary concern is that the bathtub-shape of most of the existing extended Weibull distributions are V-shapes, and therefore may not appropriately model failure time data with long-flat region bathtub-shaped FR. Many reliability problems have a bathtub-shaped which can be divide into three segments. At time zero, the early failures that may occur as a result of poor design or improper material management cause FR to significantly decline, or initial implementation issues.

The middle portion of the FR function is roughly level (constant)- which is regarded as an 'important or stable period.' The wear-out or rising period is the final component such as material degradation or fatigue. Hence, the need for new distribution that can exhibit bathtub-shaped with the long-flat region. Additive Weibull (AddW) model is one of the Weibull extended models proposed by Xie and Lai (1996) based on two Weibull reliability (RE) functions, say $S_1(x)$ and $S_2(x)$, With the following cumulative distribution function (CDF):

$$F(x) = 1 - e^{-\alpha x^\beta - \gamma x^\lambda} \quad (1)$$

where $\alpha, \gamma > 0$ are scale parameters, and $\beta > 0$ and $\lambda > 0$ are the model shape parameters.

The RE function to the CDF (1) is

$S(x) = S_1(x)S_2(x) = e^{-\alpha x^\beta - \gamma x^\lambda}$. The AddW has a failure rate (FR) function with bathtub-shaped additional to its parent distribution's monotone FR shapes. Some properties of the AddW model and defined the model parameters in more general forms was (Lemonte et al., 2014). They observed that the support of the parameters could be

considered as $\alpha, \gamma > 0$ and $\beta > \lambda > 0$. The AddW was recently extended to a five-parameter distribution called exponentiated additive Weibull (EAddW) distribution by (Abd EL-Baset and Ghazal 2020). The EAddW is more prosperous and flexible with one more FR (unimodal) shape compared to the AddW distribution. Also, it features a bathtub-shaped FR with a lengthy, level area essential in a reliability context (Bebbington *et al.*, 2012). The EAddW's CDF is provided by;

$$F(x) = \left(1 - e^{-\alpha x^\beta - \gamma x^\lambda}\right)^\theta \quad (2)$$

where, $\alpha, \theta, \gamma > 0$ and $\beta > \lambda > 0$.

In this paper, motivated by the growing accessibility of lifetime data with a wide range of attributes, we shall consider extending the AddW distribution from another approach to define a five-parameter model known as generalized extended additive Weibull (GExAddW) distribution. We construct the new model via re-defining the AddW model to a three-parameter AddW based on two alternate parametric forms of the Weibull RE

functions, $S_1(x) = e^{-x^\beta}$ and $S_2(x) = e^{-\left(\frac{x}{\gamma}\right)^\lambda}$, we

add two extra positive shape parameters to the resulting model. The approach is important as it provides a parametric class of distributions with greater flexibility to analyze monotone and non-monotone FR functions (Shakhathreh *et al.*, 2020). The proposed distribution can cope not just the four conventional FR functions (decreasing, increasing, bathtub, and unimodal) but also with modified unimodal (increasing-decreasing-increasing) FR and decreasing-increasing-decreasing FR functions that some of the generalizations of either the Weibull or AddW cannot fit appropriately. It is necessary to mention that the main goal of extending the AddW model is to define a distribution with modified bathtub-shaped FR functions Shakhathreh *et al.*, (2020), and long-flat region bathtub-shaped FR function. Mohammed and Ugwuowo (2021) studied and developed another distribution with varying failure rate shapes. The exponentiated new weighted Weibull distribution was developed by Mohammed and Damisa (2018), the characterization and

2.0 DEFINITION OF THE DISTRIBUTION

Definition 2.1: The standard Weibull distribution's RE function is provided by;

$$S(x) = e^{-\left(\frac{x}{\gamma}\right)^\lambda}, x > 0 \quad (3)$$

Since the CDF(1) can be expressed as

applications of a three-parameter improved Weibull-Weibull distribution was studied by (Mohammed *et al.*, 2024). Mohammed and Ugwuowo (2020) extended Weibull distribution using a two-parameter family of distributions. Bayesian and classical estimations of enhanced Weibull-Weibull distribution for survival times data by (Wang *et al.*, 2022). Another compound Weibull distribution with application to failure time data by (Mohammed and Ugwuowo, 2020). Thus, in this paper, we study the five-parameters GExAddW distribution, which extends many classical lifetime distributions including Weibull, Exponential, Rayleigh distributions. It also generalized some well-known extensions of Weibull distribution, including exponentiated Weibull, exponential-Weibull, exponential Weibull distributions, and many others. A sturdy failure time model in the shape of a bathtub for a two-component system, suitable for complete and censored reliability data (Abba *et al.*, 2023). Abba and Wang (2024) studied a novel failure times model for systems with one or two failure modes A Bayesian investigation utilizing Hamiltonian Monte Carlo modeling. A reliable multi-risk model and its significance for reliability: a Bayes analysis using Hamiltonian Monte Carlo techniques was studied by (Abba *et al.*, 2024). A survival and reliability model for systems with one or two failure modes, with applications to survival datasets (Abba *et al.*, 2022) and application of a novel, adaptable four-parameter bathtub curve failure rate model to data with right-censoring was carried out by (Al-Essa *et al.*, 2023).

The GExAddW distribution is proposed with the hope that it can model lifetime data from many real-world scenarios more accurately than other lifetime distributions, particularly reliability engineering. The model has a friendly and handy structure and has numerical capabilities with modern computational software availability. We present a thorough description of the distribution properties. Furthermore, we take into consideration various estimation techniques for determining the parameters of the GExAddW distribution.

$F(x) = 1 - S_1(x)S_2(x)$ then letting $S_2(x)$ to be given by Equation(3) and $\alpha = 1$ in $S_1(x)$ we can have the CDF of the three-parameters AddW distribution given by;

$$F(x) = 1 - e^{-x^\beta - \gamma x^\lambda} \quad (4)$$

where, $\gamma, \beta, \lambda > 0$. Thus, the CDF (1) is modified to a three-parameters AddW distribution with RE function, $S(x)$ as a product of the two Weibull RE functions, each with an alternate parametric form, that is $S(x) = S_1(x) S_2(x)$.

Definition 2.2: The extended additive Weibull (ExAddW) distribution's CDF can be read as;

$$F(x) = 1 - e^{-\left(x^\beta - \left(\frac{x}{\gamma}\right)^\lambda\right)^k} \quad (5)$$

where, $x, \gamma, k > 0$ and $\beta, \lambda \in (0, \infty)$. Equation (5) extends the CDF of the AddW model in Equation (4). The notation ExAddW ($\gamma, k, \beta, \lambda$) is adopted for a given random variable, X , with (5). Differentiating (5) with respect to x , we have the probability density function (PDF) of X as;

$$f(x) = \frac{k \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma}\right)^{\lambda-1} \right)}{\left(x^\beta + \left(\frac{x}{\gamma}\right)^\lambda \right)^{1-k} e^{\left(x^\beta - \left(\frac{x}{\gamma}\right)^\lambda \right)^k}} \quad (6)$$

Exponentiating the CDF (5) with $v > 0$, the GExAddW distribution's CDF is derived as;

$$F(x) = \left[1 - e^{-\left(x^\beta - \left(\frac{x}{\gamma}\right)^\lambda \right)^k} \right]^v \quad (7)$$

where, $\gamma, k, v > 0$ and $\beta, \lambda \in (0, \infty)$. We adopted the notation X follows GExAddW ($\gamma, k, v, \beta, \lambda$) with the function in (7). The PDF has the form;

$$f(x) = \frac{kv \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma}\right)^{\lambda-1} \right) Y_1(x)}{\left(x^\beta + \left(\frac{x}{\gamma}\right)^\lambda \right)^{1-k} (1 - Y_1(x))^{1-v}} \quad (8)$$

where, $Y_1(x) = e^{\left(x^\beta - \left(\frac{x}{\gamma}\right)^\lambda \right)^k}$ is considered as the RE function of the ExAddW. The FR and reversed FR (RFR) functions to the PDF (8), respectively, takes the following structure;

$$h(x) = \frac{kv \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma}\right)^{\lambda-1} \right) Y_1(x)}{\left(x^\beta + \left(\frac{x}{\gamma}\right)^\lambda \right)^{1-k} \left((1 - Y_1(x))^{1-v} + Y_1(x) - 1 \right)} \quad (9)$$

and

$$r(x) = \frac{kv \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma}\right)^{\lambda-1} \right) Y_1(x)}{\left(x^\beta + \left(\frac{x}{\gamma}\right)^\lambda \right)^{1-k} (1 - Y_1(x))} \quad (10)$$

where, $\gamma, k, v > 0$ and $\beta, \lambda \in (0, \infty)$. Note that, $r(x)$ is proportional to the RFR of the ExAddW ($r_{\text{ExAddW}}(x)$) with proportionality constant v , that is, $r(x) = v r_{\text{ExAddW}}(x)$. Supposing we take

$\beta = \lambda^{-1}$. in Equation (7), we can yet have another sufficiently flexible model by Lemonte et al., (2014) that can be used in practice with a smaller number of parameter than the GExAddW distribution. The resulting model can be called generalized extended reduced additive Weibull (GExRAddW) distribution with FR function given by;

$$h(x) = \frac{kv \left(\lambda^{-1} x^{\frac{1-\lambda}{\lambda}} + \lambda \gamma^{-1} \left(\frac{x}{\gamma}\right)^{\lambda-1} \right) Y_2(x)}{\left(x^{\frac{1}{\lambda}} + \left(\frac{x}{\gamma}\right)^\lambda \right)^{1-k} \left((1 - Y_2(x))^{1-v} + Y_2(x) - 1 \right)} \quad (11)$$

where, $\gamma, k, v > 0$, $\lambda > 1$ and

$$Y_2(x) = e^{\left(x^{\lambda-1} + \left(\frac{x}{\gamma}\right)^\lambda \right)^k}$$

As a family of distributions, the GExAddW contain special models as listed in Table 1. Several renowned distributions are included, including exponentiated Weibull by Mudholkar and Srivastava (1993) generalized linear failure rate by Sarhan and Kundu (2009) and exponential Weibull distributions by (Cordeiro et al., 2014). Where RAddW stands for reduced Additive Weibull, $\alpha = \lambda^{-1}$ and $\theta = \gamma^{-\lambda}$.

Table 1: Sub-models of the GExAddW distribution

β	λ	γ	κ	ν	Distribution
0	-	-	1	-	Exponentiated Weibull (ExW)
0	-	-	1	1	Weibull (W)
-	1	-	-	-	Generalized extended exponential Weibull (GExEW)
-	1	-	-	1	Extended exponential Weibull (ExEW)
-	1	-	1	1	Exponential Weibull (EW)
-	-	-	-	1	Extended Additive Weibull (ExAddW)
α	-	-	-	-	Generalized Extended RAddW (GExRAddW)
-	-	-	1	1	Additive Weibull (AddW)
-	-	-	1	-	Exponentiated Additive Weibull (EAddW)
-	1	-	1	-	Modified generalized linear failure rate (MGLFR)
2	1	-	1	-	Generalized linear failure rate (GLFR)
0	1	-	1	-	Exponential exponential (EE)
0	1	-	1	1	Exponential (E)
0	2	θ	1	-	Generalized Rayleigh (GR)
0	2	θ	1	1	Rayleigh (R)

3.0 STATISTICAL PROPERTIES OF THE DISTRIBUTION

3.1 Asymptotes and shapes of the PDF and FR functions

Here, we determine the asymptotes of the density in (8) and the derived FR (9) function of the GExAddW distribution, respectively. For a relatively small Here, we determine the asymptotes of the derived and simplified PDF (8) and FR (9) function of the GExAddW distribution, respectively. For a relatively small $x(x \rightarrow 0)$ and $\beta = 1$, we have;

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = \begin{cases} \kappa \nu x^{\kappa \nu - 1}, & \lambda > 1 \\ \frac{\lambda \kappa \nu}{\gamma^{\lambda \kappa \nu}} x^{\lambda \kappa \nu - 1}, & \lambda < 1 \end{cases}$$

and for a big enough $x(x \rightarrow \infty)$ and $\beta = 1$, the asymptotes of the (8) and (9) are

$$\lim_{x \rightarrow \infty} f(x) = \begin{cases} \frac{\lambda \kappa \nu}{\gamma^{\lambda \kappa \nu} x^{1 - \kappa \lambda}} \exp\{-(x/\gamma)^{\kappa \lambda}\}, & \lambda > 1 \\ \kappa \nu x^{\kappa - 1} \exp\{-x^{\kappa}\}, & \lambda < 1 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} h(x) = \begin{cases} \frac{\lambda \kappa}{\gamma^{\lambda \kappa} x^{1 - \kappa \lambda}}, & \lambda > 1 \\ \kappa x^{\kappa - 1}, & \lambda < 1 \end{cases}$$

Some possible shapes of the PDF (8) are presented in Figure 1, which reveals that the density function have a decreasing, and at least a unimodal shape. As shown in Figures 2 and 3, the FR function (9) has six different FR functions, including the four standard FR shapes, the modified bathtub-shaped (increasing-decreasing-increasing) and decreasing-increasing-decreasing FR functions. It is important stressing that the FR function of the GExAddW distribution can exhibits a bathtub-shaped with or without long-constant (long-flat) region. We can observe from Shakhatareh *et al.*, (2019) that the FR shapes of most of the existing lifetime distributions are V-shaped with no constant middle parts, which made it difficult for the distributions to model data sets with long-flat regions. Hence, the GExAddW model could be a good candidate to provide better fit to data set with such type of bathtub-shaped FR.

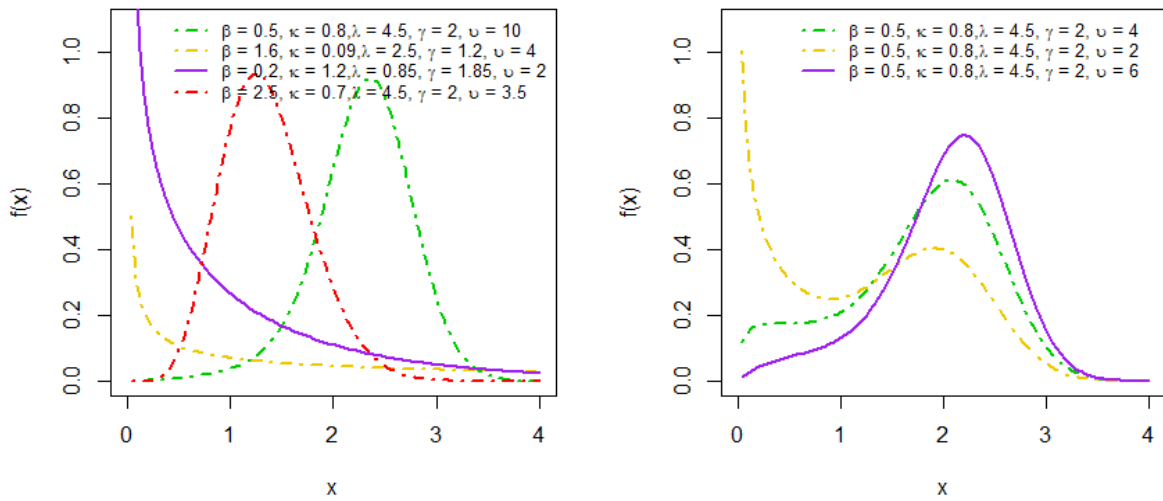


Figure 1. Various shapes of the PDF for different values of the parameters.

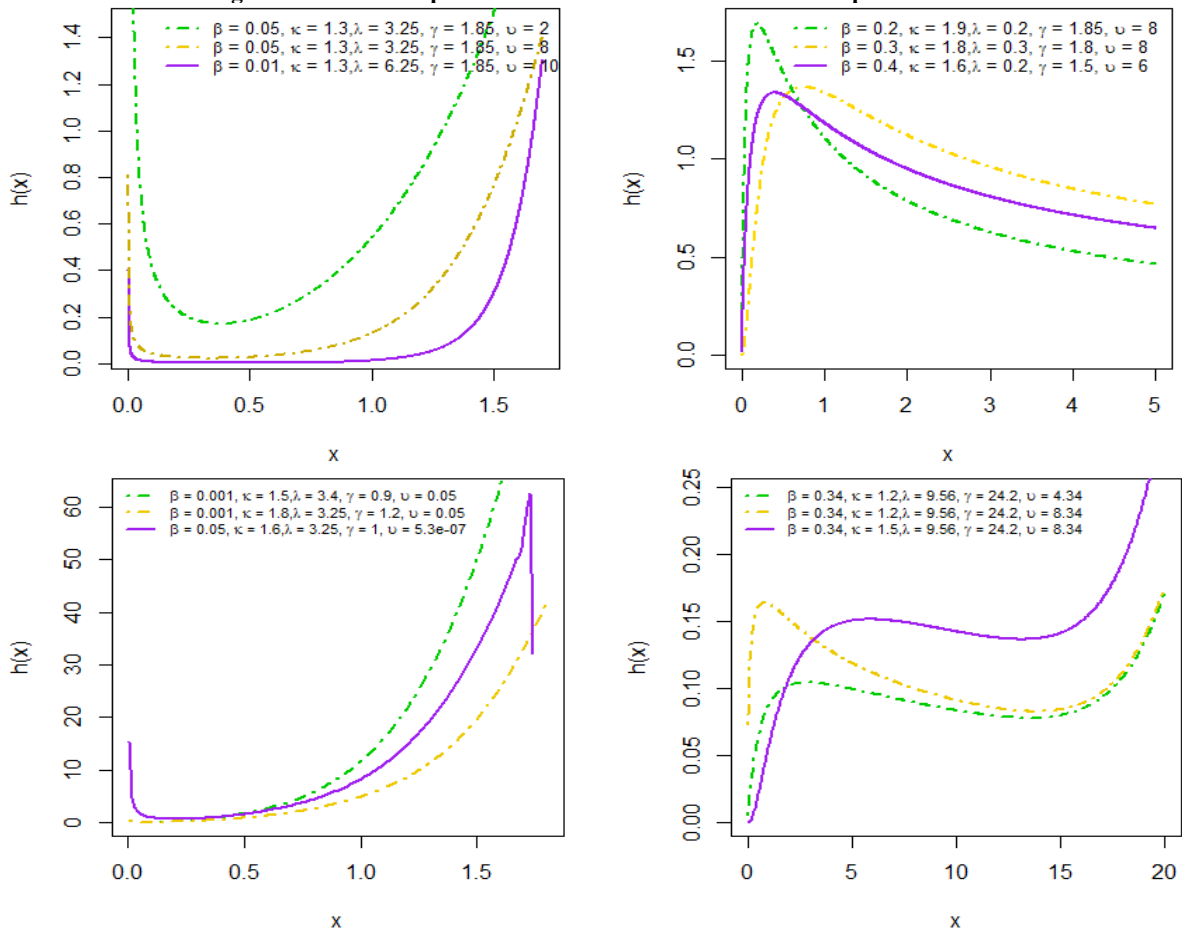


Figure 2. Various plots of the FR function with distinct shapes at distinct parameter values.

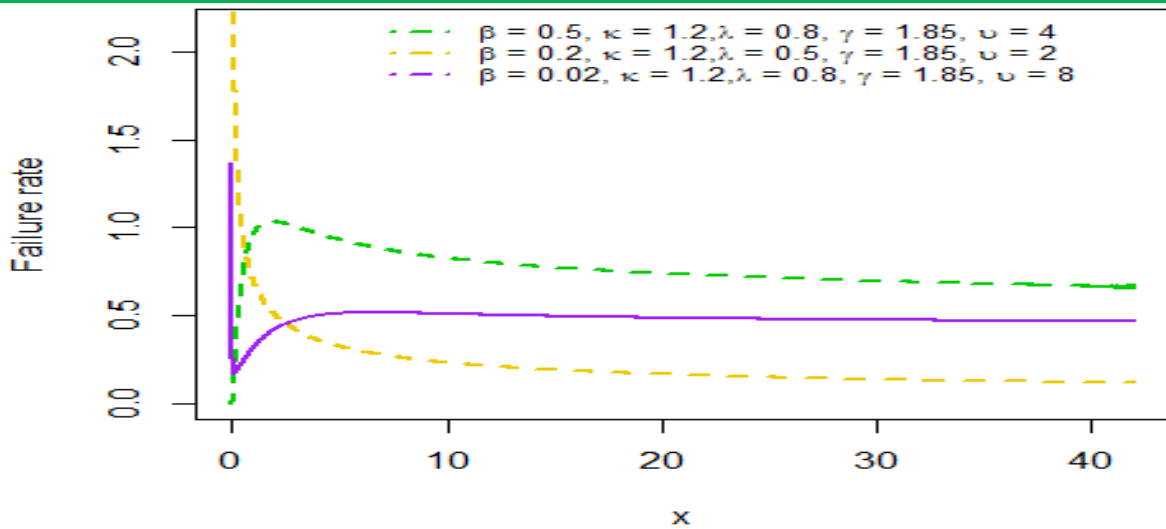


Figure 3. plot of the FR function with distinct shapes at distinct parameter values.

3.2 Quantile Function of the distribution

For a random variable $X \sim GExAddW(\beta, \lambda, \gamma, k, \nu)$ the quantile function, x_q ($0 < q < 1$) of X , is the solution to the nonlinear equation that follows.

$$\left(x^\beta + \left(\frac{x}{\gamma} \right)^\lambda \right)^k + \log(1 - q^{1/\nu}) = 0 \tag{12}$$

Numerical methods can be used to determine the quantile, since Equation (12) lacks a closed-form solution in x_q . Thus, we provide the algorithm for simulating data from $GExAddW(\beta, \lambda, \gamma, k, \nu)$ model as follows.

- i. Generate q_i from uniform distribution, that is, $q_i \sim U(0,1)$, $i = 1, 2, \dots, n$, and thereafter,

- ii. Evaluate $\left(x_i^\beta + \left(\frac{x_i}{\gamma} \right)^\lambda \right)^k + \log(1 - q_i^{1/\nu}) = 0$ (13)

To check the consistency of the sample drawn from the $GExAddW(\beta, \lambda, \gamma, k, \nu)$ distribution using Equation (13), we provide in Figure 4, some histograms for three distinct samples at various parameters values. The plots indicate the consistency of the distribution with the simulated data.

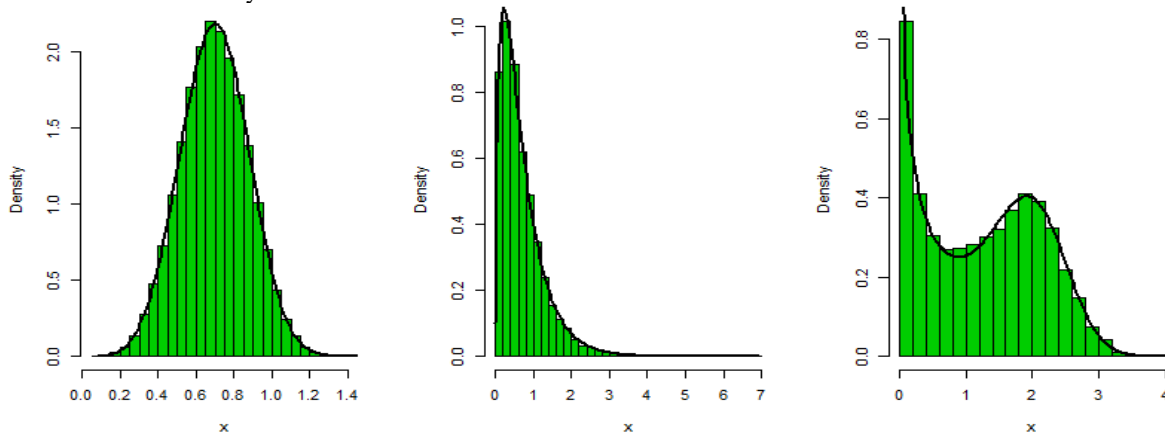


Figure 4. Histograms of the simulated samples: left-panel- $\beta=1.65, \lambda=1.5, \gamma=1.2, \kappa=2.2$, and $\nu=1.5$, center-panel- $\beta=1, \lambda=1, \gamma=1.2, \kappa=1$, and $\nu=1.5$, and right-panel- $\beta=0.5, \lambda=54.5, \gamma=2, \kappa=0.8$, and $\nu=2$

3.3 The r^{th} Moment of the $GExAddW(\beta, \lambda, \gamma, k, \nu)$

The r^{th} Moment of the $GExAddW(\beta, \lambda, \gamma, k, \nu)$ is given by;

$$E(x^r) = \sum_{j=0}^{\infty} (-1)^j \binom{\nu-1}{j} \int_0^{\infty} x^r k \nu \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda-1} \right) \left(x^{\beta} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda} \right)^{k-1} e^{-\left(j+1 \right) \left(x^{\beta} + \left(\frac{x}{\gamma} \right)^{\lambda} \right)^k} dx \quad (14)$$

Equation (14) lacks a close-form solution. To determine a numerical solution to this integral, a numerical method is required.

3.4 Order Statistics of the Distribution

Suppose X_1, \dots, X_n is a random sample from the $GExAddW(\beta, \lambda, \gamma, k, \nu)$ and let $X_{1:n} < \dots < X_{i:n}$ The pdf, $f_{i:n}(x)$ of the i^{th} order statistic is obtainable by; indicate the corresponding order statistics obtained from this sample.

$$f_{i:n}(x) = \frac{n!k\nu}{(i-1)!(n-i)!} \sum_{j,m,p=0}^{\infty} \psi_{j,m,p} \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda-1} \right) \left(x^{\beta} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda} \right)^{k(1+p)-1} \quad (15)$$

$$\text{where, } \psi_{j,m,p} = \frac{(-1)^{j+m+p}}{p!} \binom{n-i}{j} \binom{\nu(i+j)-1}{m} (m+1)^p.$$

Consequently, the PDF of the $GExAddW(\beta, \lambda, \gamma, k, \nu)$'s minimum and maximum order statistics are respectively given by;

$$f_{1:n}(x) = nk\nu \sum_{j,m,p=0}^{\infty} \Theta_{j,m,p} \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda-1} \right) \left(x^{\beta} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda} \right)^{k(1+p)-1}$$

$$\text{where, } \Theta_{j,m,p} = \frac{(-1)^{j+m+p}}{p!} \binom{n-1}{j} \binom{\nu(1+j)-1}{m} (m+1)^p \text{ and}$$

$$f_{n:n}(x) = nk\nu \sum_{j,p=0}^{\infty} \frac{(-1)^{j+p}}{p!} \binom{\nu n-1}{j} (j+1)^p \left(\beta x^{\beta-1} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda-1} \right) \left(x^{\beta} + \lambda \gamma^{-1} \left(\frac{x}{\gamma} \right)^{\lambda} \right)^{k(1+p)-1}$$

4.0 APPLICATION

The results of the estimated parameters for the GExAddW, EMWE, MW, EW, AddW, EAddW, and GExtEW distributions for fitting the Aarset datasets are shown and discussed in the next two subsections. Plotting the survival functions, FRFs, box-plots, and P-P plots of the trained models helps to visualize the optimum parameters as well.

4.1 Failure Times of Device

The dataset provided below shows the times to failure of fifty electronic items that were put through a life test at time zero. The dataset, which was first described by Aarset (1987) has been widely regarded by other scholars as the standard for fitting and evaluating lifetime distributions with bathtub-shaped FR. The GExtEW distribution by Shakhathreh *et al.*, (2020) the EAddW distribution by Abd EL-Baset

and Ghazal (2020) and the expansion of the Chen distribution by Tarvirdizade and Ahmadpour (2021) are some of the most recent publications that have made use of the data. Since the Aarset dataset is known to contain a single failure mode, the proposed distribution can be used to analyze it.

The lifetime of 50 electronic devices.

0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

The optimized parameter values (MLEs) are presented in Table 2, In Table 3, AIC, BIC, and KS for the competitive distributions were displayed. Table 3 shows that compared to other trained models in the study, the five-parameter GExAddW model

has the lowest values of AIC, BIC, and KS and the greatest p-value of 0.5843. This indicates that the

model is better suited for the Aarset dataset.

Table 2: MLEs of the GExAddW distribution's parameters as well as those of other models

Models	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$
MW	0.023	0.063			0.3600
EW	4.671	0.011			0.1450
GExAddW	0.0266	6.1566	63.927	8.7177	89.138
AddW	1.622×10^{-8}	4.134	0.09	0.4610	
EMWE	3.165	7.02×10^{-5}	49.206		0.1440
EAddW	5.368×10^{-15}	7.541	4.902	0.056	345.18
GExEW	0.088	34.029	0.508	0.003	0.1280

Table3: Performance of the distributions

Models	AIC	BIC	KS(P-value)
MW	460.32	466.047	0.134 (0.328)
EW	464.272	470.008	0.206 (0.0291)
GExAddW	426.37	435.9302	0.110 (0.5843)
AddW	451.427	459.075	0.126 (0.4073)
EMWE	435.632	443.28	0.140 (0.2832)
EAddW	441.711	451.271	0.119 (0.4783)
GExEW	453.173	462.733	0.151 (0.2035)

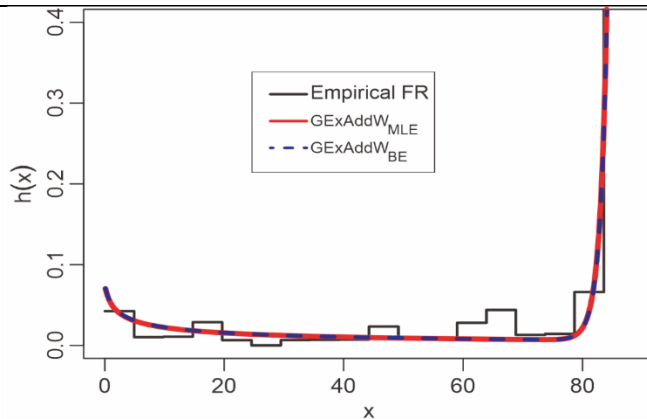


Figure 5. plots for the fitted GExAddW distribution and other distributions for Aarset dataset

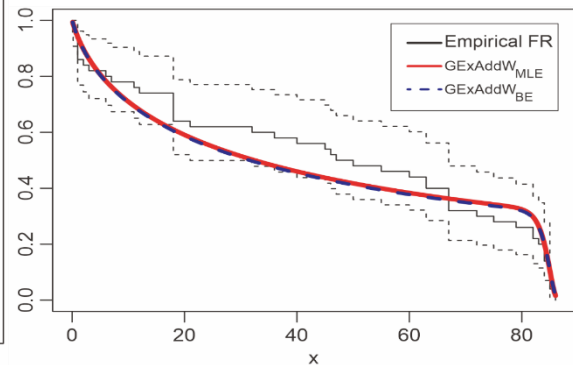


Figure 6. plots for the fitted GExAddW distribution and other distributions for Aarset dataset

Table 2: Posterior summaries of the (Hamiltonian Monte Carlo) HMC method: 3 parallel chains, each with 2000 iterations; warm up = 1000; post-warm up draws per chain=1000, total post-warm up draws=3000. Aarset data

Parameters	Posterior mean	SD	Median	95% Credible interval
γ	0.0062	0.0007	0.0061	[0.0057, 0.0075]
α	6.1444	0.6861	6.1006	[4.9037, 7.5768]
τ	62.68	3.8094	62.576	[55.589, 70.466]
β	91.539	0.4021	91.53	[90.780, 92.363]
λ	37.653	3.8308	37.527	[30.501, 45.698]

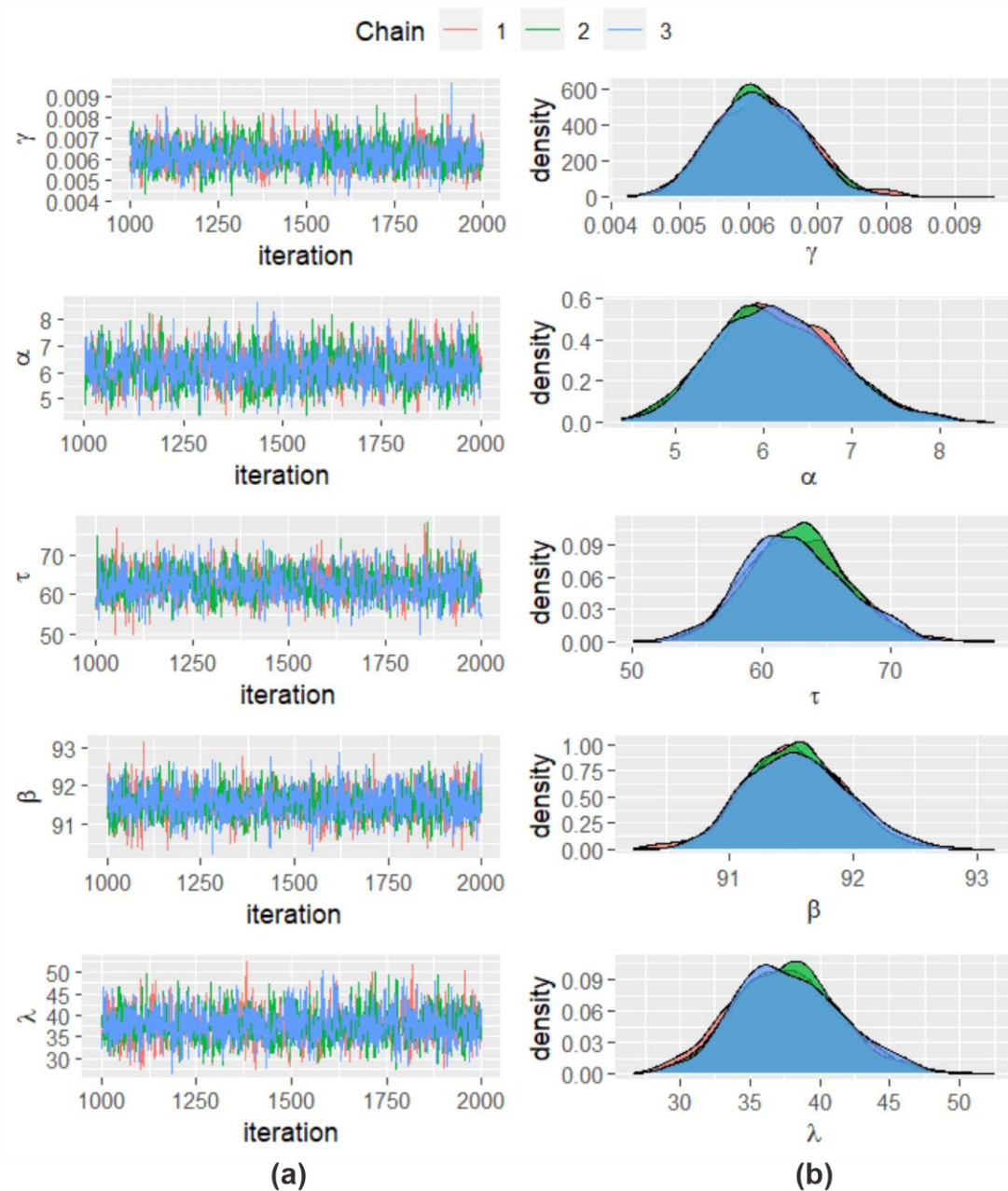


Figure 7. Trace plots (a) and (b) marginal posterior densities of the GExAddW parameters for the Aarset dataset under Cauchy prior

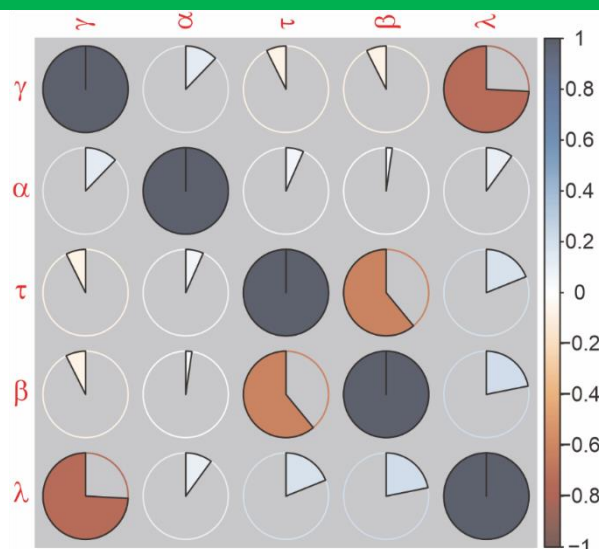


Figure 8. Plot of the correlation matrix for the posterior samples obtained from the HMC methodology

5.0 CONCLUSION

The new distribution is constructed by re-defining the AddW model to a three-parameter AddW based on two alternate parametric forms of the Weibull reliability functions and two extra positive shape parameters were added to the resulting model to form the generalized extended additive Weibull (GExAddW) distribution. Some distributional properties were presented, including the quantile function, moments and failure rate (FR) function. Bayesian inference and the maximum likelihood method are considered for estimating the GExAddW parameters. We validated the performance of the GExAddW model using one times to failure of devices data characterized with bathtub-shaped FR. On the basis of the data FR characterization, a comparison study was conducted with six bathtub-shaped methodologies: the EW, MW, AddW, EMWE, EAddW, and GExEW distributions. The model selection criteria, survival and HR curves are presented for the proposed. The numerical and pictorial results indicated that the GExAddW distribution may be the best choice for describing the failure times of the devices. Consequently, the findings recommend that the best way to extend the AddW model is to modify it while holding its physical meaning.

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